GEOMETRY OF CONTACT TRANSFORMATIONS IN DIMENSION THREE

Doctoral dissertation

by

Thomas Erlandsson

U.U.D.M. Report 1981:2

UPPSALA UNIVERSITY

DEPARTMENT OF MATHEMATICS

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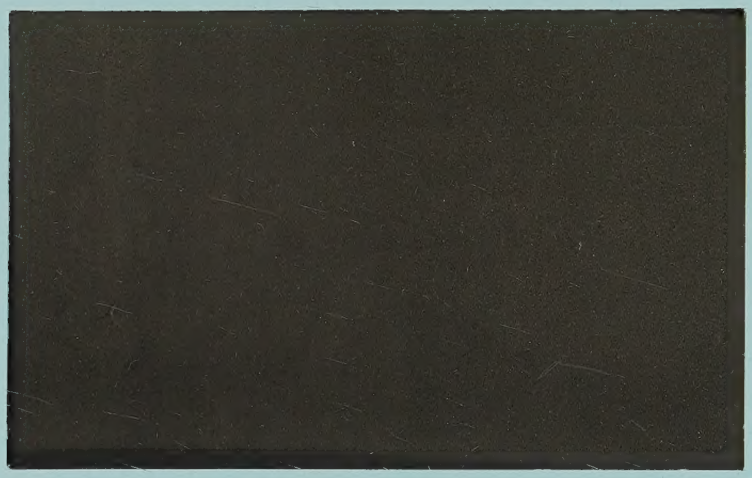

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Abstract

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It is a classical theorem that all contact structures on a manifold are equivalent locally. Using geometric arguments, we show that there are non-equivalent contact structures on the three-dimensional euclidean space. We show as a corollary, that there are contact structures on the three-dimensional torus, and on the three-dimensional sphere, which are non-equivalent for geometric reasons.

The result follows from a study of integral curves in the contact structures, and from a new Knotting Criterion. This criterion is a special case of a new Genus Estimation Theorem for orientable, embedded surfaces, with boundary embedded in space in a "right-handed" way. The theorem follows from Morse theory, and from a formula by Poincaré for the number of singularities on a closed, orientable surface.

Key Words and Phrases: Contact form, contact structure, contact transformation, equivalence and non-equivalence of contact structures, knots and links in three dimensions, surfaces, genus.

AMS Subject Classifications (1980): Primary 53C15, 57M25; Secondary 58A10, 58A30.

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To my parents

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INTRODUCTION

§1. Background

Contact transformations were introduced by Sophus Lie. An account of his theory was published in a book by Lie in 1896 [8].

In the plane Lie considered transformations of what he called line elements. A line element in the plane is a triple (x,y,p), where x and y are the coordinates of a point in the plane, and p determines a line through the point.

Lie observed that many important transformations in the plane are transformations of line elements, and that they have a special geometric property. The direction, determined by a tangent to a curve at a point, is transformed to the direction determined by the tangent to the transformed curve at the image point. (Fig. 1)

Fig. 1

So these transformations can be said to preserve <u>contact</u> between a curve and its tangent at a point, and Lie therefore called them <u>contact transformations</u>.

Lie also considered transformations of surface elements in space. A surface element is a quintuple (x,y,z,p,q), where x, y and z are the coordinates of a point in space, and p and q determine a plane through the point. Lie's book contains an introduction to his theory of contact transformations of surface elements, i.e. of transformations which preserve contact between a surface and its tangent plane at a point. Lie planned to give a full account of his theory in a second volume, but it was never published.

The contact condition can be written as a differential relation. In the plane we have

$$dy_1 - p_1 dx_1 = \lambda(dy - pdx), \tag{1}$$

where $x_1 = x_1(x,y,p)$, $y_1 = y_1(x,y,p)$, $p_1 = p_1(x,y,p)$ define a transformation, and $\lambda = \lambda(x,y,p)$ is a function, non-zero everywhere.

Lie then considers x, y and p as coordinates of a point in space, so the theory of line elements in the plane becomes a geometry of the three-dimensional space. The relation (1) then means, that every contact transformation in the plane corresponds to a point transformation in space, under which the equation $dy - pdx = 0$ remains invariant.

Lie finally replaces the form $dy - pdx$ by a suitable class of forms $Xdx + Ydy + Zdz$. To each form corresponds a geometry, which is called a contact structure.

§2. The standard contact structure

A contact form on an orientable, $(2n+1)$-dimensional, differentiable manifold M is a 1-form ω such that $\omega \wedge (d\omega)^n \neq 0$ at each point. A contact structure on M is a field of $2n$-dimensional hyperplanes, determined by an equation $\omega = 0$, where ω is a contact form.

We shall mainly study contact structures in the euclidean space, R^3. We give examples of contact structures in certain 3-manifolds in §7. For examples of contact structures in $(2n+1)$-dimensional manifolds, see [2].

In R^3 we have the following explicit definitions of contact form and contact structure.

Let $X(x,y,z)dx + Y(x,y,z)dy + Z(x,y,z)dz$ be a form in R^3. It is a contact form if $X(Y_z - Z_y) + Y(Z_x - X_z) + Z(X_y - Y_x) \neq 0$.

The functions X, Y and Z in the form $Xdx + Ydy + Zdz$ determine a plane with normal vector (X,Y,Z) at each point of space. These planes are called a contact structure, and each plane is called a contact plane.

A contact structure does not change if we multiply the corresponding form by a function which is non-zero everywhere. We therefore define a contact structure by an equation $Xdx + Ydy + Zdz = 0$.

We shall now describe three contact structures in R^3. We refer to any of them as the standard contact structure. The reason is that they are all equivalent in a way we shall describe in §3.

EXAMPLE 1 The contact structure $xdy + dz = 0$ is illustrated in figure 2. The contact planes rotate along lines parallel to the x-axis. The angle of rotation measured from the y-axis is less than 90°.

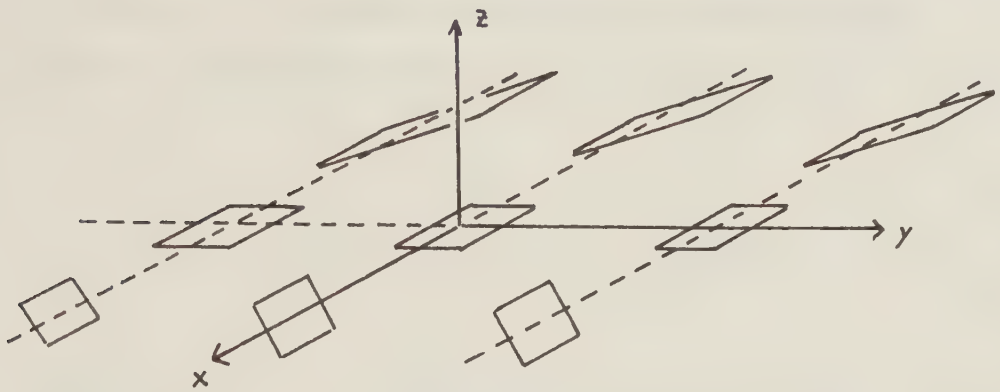


Fig. 2

EXAMPLE 2 The equation $\sin x dy + \cos x dz = 0$ defines also a standard contact structure, and is illustrated in figure 3. The contact planes rotate along lines parallel to the x-axis as in the previous example, but the planes rotate an infinite number of times.

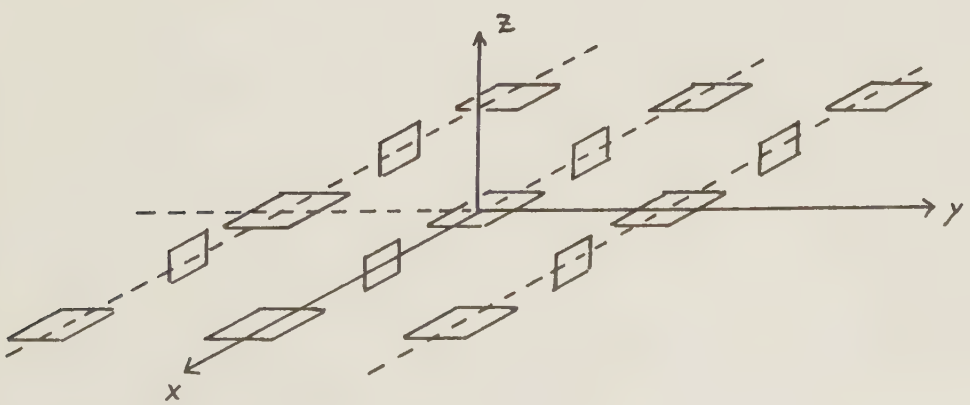


Fig. 3

EXAMPLE 3 Another standard contact structure is defined by $xdy - ydx + dz = 0$, and is illustrated in figure 4. It has cylinder symmetry. It can be defined in cylinder coordinates by $r^2 d\theta + dz = 0$. The contact planes rotate along the radial lines from the z-axis which are parallel to the xy-plane. The angle of rotation from the z-axis is less than 90°.

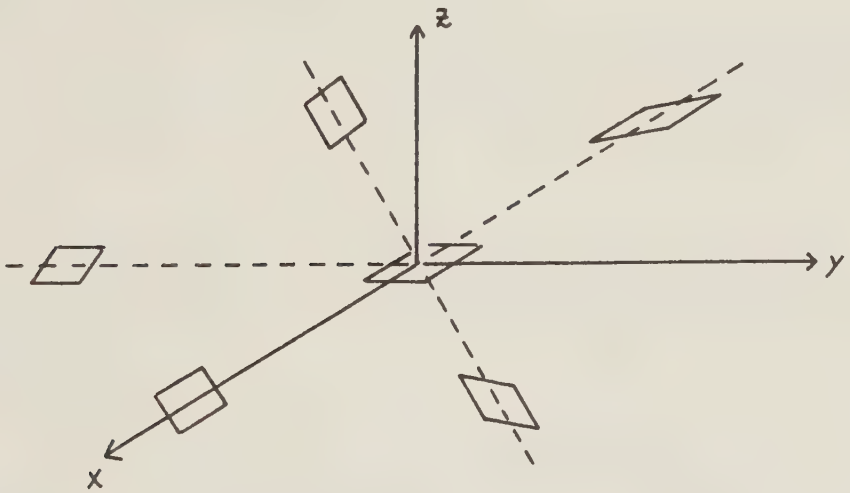


Fig. 4

§3. Equivalence of contact structures

We define contact transformations and equivalence of contact structures in orientable, $(2n+1)$-dimensional, differentiable manifolds M. Contact transformations must be C^2, i.e. continuously differentiable twice. We assume that all transformations are C^2, and we assume that all manifolds, functions and forms are sufficiently regular to give C^2-transformations.

Let ω be a 1-form, and $f: M \to M$ a differentiable transformation of M. The form ω is transformed under f to a 1-form, which we denote by $f^*\omega$.

A differentiable, one-to-one map f of M onto itself, such that the inverse function f^{-1} is also differentiable, will be called a diffeomorphism.

Let now ω_0 and ω_1 be two contact forms. Assume there is a function λ, non-zero everywhere, and a diffeomorphism $f: M \to M$, such that

$$f^*\omega_1 = \lambda\omega_0. \tag{2}$$

Such a diffeomorphism will be called a contact transformation.

Two contact structures, defined by $\omega_0 = 0$ and $\omega_1 = 0$ resp., are said to be equivalent, when the relation (2) is satisfied for a diffeomorphism f and some function λ, which is non-zero everywhere. We say that the two forms ω_0 and ω_1 are equivalent when $\lambda \equiv 1$.

In the euclidean space R^3 we shall let ω_0 be the standard form $xdy + dz$. The equivalence condition is then given in the following way.

A contact structure in R^3, defined by $Xdx + Ydy + Zdz = 0$, is equivalent to the standard structure, defined by $xdy + dz = 0$, when the relation (3) below is satisfied for a diffeomorphism ϕ, φ, ψ of R^3, and some function λ, non-zero everywhere.

$$Xdx + Ydy + Zdz = \lambda(\phi d\varphi + d\psi). \qquad (3)$$

We say that the two forms are equivalent when $\lambda \equiv 1$ in (3).

EXAMPLE 4 The form $sinxdy + cosxdz$ in example 2 is equivalent to the standard form $xdy + dz$. We have $sinxdy + cosxdz = \phi d\varphi + d\psi$, where $\phi = -ycosx + zsinx$, $\varphi = x$, $\psi = ysinx + zcosx$.

EXAMPLE 5 The form $xdy - ydx + dz$ in example 3 is also equivalent to the standard form $xdy + dz$. We have $xdy - ydx + dz = \phi d\varphi + d\psi$, where $\phi = 2x$, $\varphi = y$, $\psi = z - xy$.

We have found that the three contact forms $xdy + dz$, $sinxdy + cosxdz$ and $xdy - ydx + dz$ are all equivalent. We therefore refer to any of them as the standard contact form in R^3. It follows that the corresponding contact structures are equivalent too. We therefore refer to any of them as the standard contact structure in R^3.

REMARK When we consider contact structures with cylinder symmetry, defined in cylinder coordinates by $Fdr + Gd\vartheta + Hdz = 0$, we let ω_0 be the standard form $r^2d\vartheta + dz$. The relation (3) is then replaced by

$$Fdr + Gd\vartheta + Hdz = \lambda(R^2d\theta + d\psi), \qquad (4)$$

where R, θ and ψ define a diffeomorphism.

It is a classical theorem, that for any contact forms ω_0 and ω_1 on a manifold M, there is <u>locally</u> a diffeomorphism f such that

$$f^*\omega_1 = \omega_0.$$

Lie proves this theorem for three dimensions in [8, pp. 199-201]. For the general case, see for example [4, pp. 118-120].

<u>REMARK</u> It is not necessary to introduce the function λ locally, but it is needed in general for transformations which are not local. One obvious reason is that the volume of M with respect to $\omega \wedge (d\omega)^n$ is invariant. In Chapter 1 we discuss the function λ, and we give examples of contact forms in three dimensions, where the associated volumes are the same, but where we must have $\lambda \not\equiv 1$ in order to obtain equivalence.

The following question arises naturally: <u>Are all contact structures on a manifold equivalent</u>?

Lie studied equivalence of contact structures under <u>projective transformations</u> of R^3. A projective transformation of the <u>plane</u> R^2 is described in figure 5. A projective transformation of the <u>space</u> R^3 is obtained in the same way, as a projection between two copies of R^3 in the 4-dimensional space R^4.

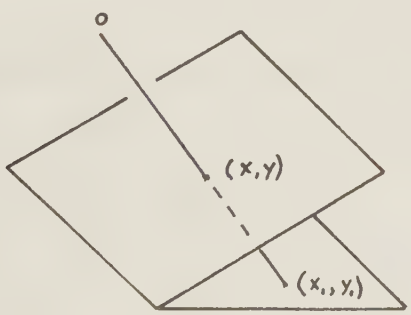


Fig. 5

A projective transformation of space is given by

$$\begin{cases} x_1 = \dfrac{a_1x + b_1y + c_1z + d_1}{a_4x + b_4y + c_4z + d_4}, \\ y_1 = \dfrac{a_2x + b_2y + c_2z + d_2}{a_4x + b_4y + c_4z + d_4}, \\ z_1 = \dfrac{a_3x + b_3y + c_3z + d_3}{a_4x + b_4y + c_4z + d_4}, \end{cases}$$

where a_i, b_i, c_i, d_i $(i = 1, 2, 3, 4)$ are constants, whose determinant is non-zero.

A projective transformation maps lines to lines, and planes are mapped to planes.

Lie proved the following non-equivalence theorem [8, pp. 218-220].

THEOREM (Lie) The contact structure, defined by $xdy - ydx + dz = 0$, is not equivalent under projective transformations to the structure defined by $xdy + dz = 0$.

Lie remarks that his theorem follows from the analytic expression of a projective transformation, but Lie also gave a geometric proof of the theorem. We shall briefly discuss Lie's geometric proof in §5, and also give a somewhat different proof in Chapter 2.

REMARK Lie observed that $xdy - ydx + dz$ is equivalent to the form $dy - pdx$ under a one-to-one transformation which is not projective [8, p. 238].

§4. Non-standard contact structures

It is the object of this paper to prove that there are contact structures in the 3-dimensional euclidean space R^3, which are not equivalent for geometric reasons to the standard contact structure.

Such a non-standard contact structure is illustrated in figure 6. It has cylinder symmetry, and can be described by the equation $r\sin r d\vartheta + \cos r dz = 0$. The contact planes rotate an infinite number of times along the radial lines from the z-axis which are parallel to the xy-plane.

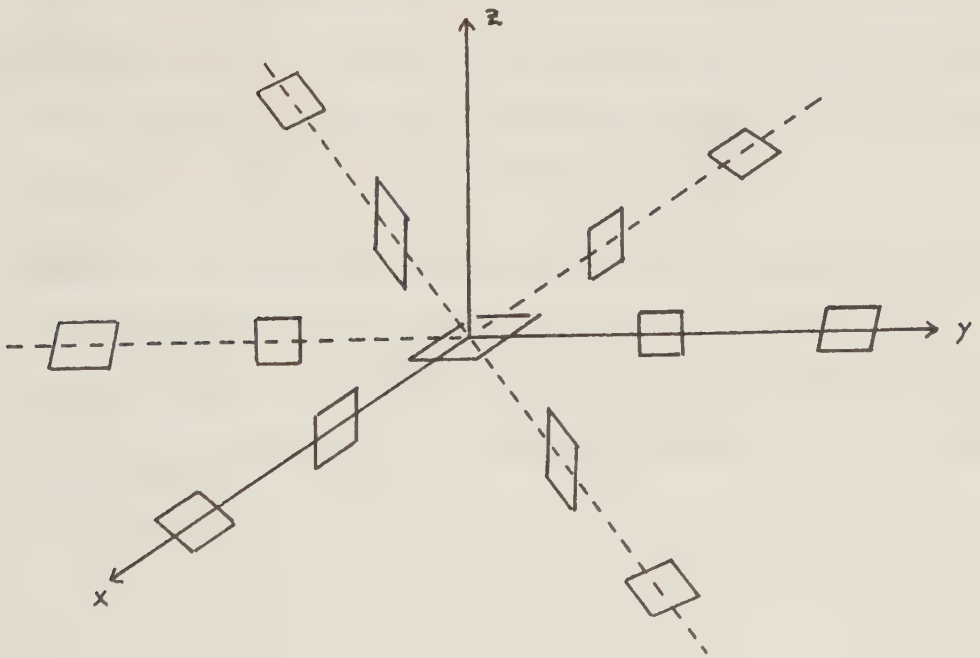


Fig. 6

Other non-standard structures can be obtained in the following way. Let n be a non-negative integer. We let the contact planes rotate $n \cdot 180^\circ$ for $0 \leq r \leq r_0$ along the radial lines from the z-axis, and for $r > r_0$ we let the planes rotate less than 180°. Near $r = 0$ we let the structure be $r^2 d\vartheta + dz = 0$. For each n we then obtain a contact structure, which we denote by C_n, and which is given by an equation $r^2 f(r) d\vartheta + g(r) dz = 0$, where $f(r) = g(r) = 1$ near $r = 0$. For $n = 0$, the planes rotate less than 180°, and then $f(r) \neq 0$. The structure for which the planes rotate an infinite number of times will be denoted by C_∞. It will follow from our proof that C_n is not equivalent to the standard structure for any $n \geq 1$, including $n = \infty$. In Chapter 1, §3, we shall show that any contact structure of the type C_0, i.e. a structure where the contact planes rotate less than 180° from the z-axis, is equivalent to the standard structure.

REMARK It is important that the non-standard structures have cylinder symmetry. The contact planes of the structure $\sin x\,dy + \cos x\,dz = 0$ rotate an infinite number of times, but the structure is equivalent to the standard structure, as we found in §3, example 4.

§5. Integral curves

Lie remarks that his non-equivalence theorem can be proved by explicit calculation [8, p. 220], but he also proved the theorem by geometric arguments.

The key to a geometric proof is the introduction of integral curves. An integral curve of a contact structure is a curve such that its tangent at each point lies in the contact plane at that point. (Fig. 7)

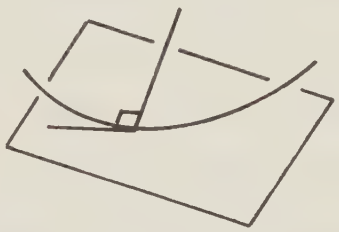


Fig. 7

An integral curve is mapped to an integral curve under a contact transformation, and as integral curves reflect the geometry of the contact structure, they can be used to detect if two contact structures are non-equivalent.

Lie observed that the structure defined by $xdy + dz = 0$ has a two-parameter family of integral lines [8, pp. 186-187], while $xdy - ydx + dz = 0$ has a three-parameter family of integral lines [8, pp. 210-211]. Therefore the two structures cannot be equivalent under projective transformations [8, pp. 218-220]. We give a somewhat different proof of Lie's theorem in Chapter 2.

Our non-equivalence theorem is the result of a study of the geometry of a disc, whose boundary is an integral curve.

§6. A new knotting criterion

Our main result, that there are non-equivalent contact structures, follows from a study of integral curves and from a knotting criterion. We discovered this knotting criterion during our study of integral curves in the standard structure. The criterion provides a quick and convenient way to decide that certain embeddings of circles in space are knotted, i.e. that they cannot bound an embedded disc. Among those are knots with trivial Alexander polynomial as we shall see in Chapter 2, §8. The criterion can be stated without reference to contact geometry, and we shall now briefly describe the criterion. We shall discuss it in detail in Chapters 2 and 3.

The basic concept in the criterion is the kinking number $\mathcal{H}$ of a regular projection of a closed curve. A projection is regular when all crossings are separated. In order to calculate $\mathcal{H}$ we give the curve an orientation as in figure 8.

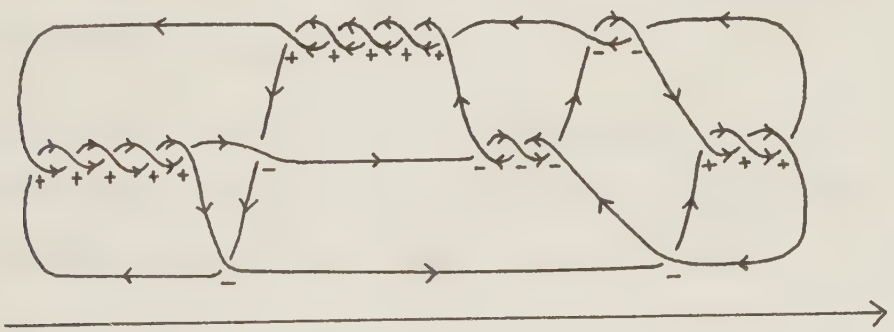


Fig. 8

Following the orientation, assign +1 to each crossing where the curve passes under itself to the left, and assign -1 to the other crossings.

The sum of these numbers is the kinking number of the projection. It is independent of the orientation of the curve. The curve in figure 8 has kinking number $\mathcal{K} = 13 - 8 = +5$.

Now fix a direction in space, and assume that all crossings twist as a right-handed screw in this direction. (Fig. 9)

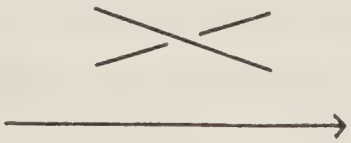


Fig. 9

Such crossings will be said to have positive twists. All crossings of the curve in figure 8 have positive twists with respect to the direction indicated in the figure.

Let b be the number of relative maxima of the position of γ in the chosen direction. Such maxima are called bridges [20]. The curve in figure 8 has two bridges, i.e. $b = 2$. We can now state our knotting criterion.

KNOTTING CRITERION Consider a closed curve embedded in space with kinking number $\mathcal{K}$ and with b bridges. Assume all crossings have positive twists. If $\mathcal{K} \geq b$, then the curve is knotted.

The curve in figure 8 has $\mathcal{K} = 5$ and $b = 2$, so the curve is knotted.

The knotting criterion is a special case of a more general Genus Estimation Theorem, which we also discovered during our study of knots embedded in the standard contact structure.

GENUS ESTIMATION THEOREM Let γ be a closed curve embedded in space with kinking number $\varkappa$ and with b bridges. Assume all crossings have positive twists. Let g be the genus of an orientable, embedded surface with boundary γ. Then

$$2g \geq \varkappa - b + 1.$$

For the curve in figure 8 we get $2g \geq 5 - 2 + 1 = 4$, i.e. $g \geq 2$. We prove the Genus Estimation Theorem in an Appendix on p. 81.

§7. Contact structures in closed 3-manifolds

Every closed, orientable 3-manifold has a contact form [9], [12], [23]. It is also known that each homotopy class of plane fields in the 3-sphere S^3, and each homotopy class in 3-manifolds covered by S^3, contains two non-equivalent contact structures [10], [11]. Contact structures in different homotopy classes in S^3, and in different homotopy classes in 3-manifolds covered by S^3, are also non-equivalent [10], [11].

These non-equivalence results are of algebraic nature. They do not follow from a study of integral curves in the contact structures. Our non-equivalence theorem is of a different type. We show that there are contact structures in R^3, which are not equivalent to the standard structure for geometric reasons.

All contact structures in R^3 are in the same homotopy class of plane fields. We can therefore use our result in R^3 to show that there are contact structures in the 3-torus T^3, and contact structures in the 3-sphere S^3, which are in the same homotopy class of plane fields, but which are not equivalent for geometric reasons.

We first consider the 3-torus T^3, which is obtained from R^3 by identifying points under the translations $x \to x + 2\pi$, $y \to y + 2\pi$, $z \to z + 2\pi$.

We observe that the contact form $\sin x\,dy + \cos x\,dz$ is invariant under these translations, so it defines a contact form in T^3. We showed in §3, example 4, that this form is equivalent to the standard form $x\,dy + dz$ in R^3.

So $\sin x\,dy + \cos x\,dz = 0$ defines a contact structure in T^3, and a contact structure in R^3, which is equivalent to the standard structure in R^3, i.e. in the covering space of T^3. We call this structure in T^3 the standard contact structure.

We shall now construct new contact structures in T^3, which are not equivalent to the standard structure.

Consider a closed curve transversal to the contact planes of the standard structure in T^3, and a thin tubular neighbourhood of this curve, i.e. a thin solid torus. Introduce the coordinates (r, ϑ, z) in the solid torus, $0 \leq r \leq r_1$. Let n be a positive integer. We construct a new contact structure in T^3 in the following way. Let the contact planes rotate $n \cdot 180^\circ$ for $0 \leq r \leq r_0$ in the solid torus, $r_0 < r_1$. For $r_0 < r \leq r_1$, we adjust the rotation to the contact structure outside the solid torus, so we get a new contact structure in T^3. This structure is not equivalent to the standard structure in T^3 for any $n \geq 1$.

We now consider the 3-sphere S^3, which can be described as those points (x_1, x_2, x_3, x_4) in the 4-dimensional space R^4, for which

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1.$$

It is easy to verify that

$$x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3 \tag{1}$$

defines a contact form in S^3. The corresponding contact structure will be called the standard contact structure in S^3.

We now construct new contact structures in S^3 with the same method we described above for T^3. For S^3, however, it does not follow immediately that the new structures are non-equivalent to the standard structure. The 3-sphere S^3 does not have R^3 as a covering space, and we must use a different procedure to show non-equivalence.

The 3-sphere S^3 can be considered as a "sphere with north pole", and there is a one-to-one correspondence between the points of S^3 outside the north pole and the euclidean space R^3 under stereographic projection.

The idea is now to use stereographic projection to map the standard contact structure in S^3 to a contact structure $\omega = 0$ in R^3. If the structure $\omega = 0$ is <u>equivalent</u> to the standard structure in R^3, then the new structures in S^3 defined above, are not equivalent to the standard structure in S^3. The reason is that our arguments for non-equivalence are such, that it is no restriction to avoid one point in S^3, for example the north pole.

If we place the north pole at $(0,0,0,1)$ in R^4, we get the following expression for the stereographic projection to the space $R^3 = (x,y,z,0)$ in R^4,

$$x = \frac{x_1}{1 - x_4}, \qquad y = \frac{x_2}{1 - x_4}, \qquad z = \frac{x_3}{1 - x_4}.$$

The form (1) is transformed under stereographic projection to a contact form $\lambda\omega$, where

$$\lambda = \frac{4}{(1 + x^2 + y^2 + z^2)^2},$$

$$\omega = (xz - y)dx + (yz + x)dy + \tfrac{1}{2}(1 + z^2 - x^2 - y^2)dz.$$

We shall discuss the forms $\lambda\omega$ and ω in more detail in Chapter 1, §4. Here we shall just give a diffeomorphism of R^3 which transforms ω to the standard form. It is convenient to use cylinder coordinates. We have

$$\omega = zrdr + r^2 d\nu + \tfrac{1}{2}(1 + z^2 - r^2)dz,$$

and the standard form is $R^2 d\theta + d\Psi$.

The diffeomorphism

$$R = r, \qquad \theta = \nu - z, \qquad \Psi = \frac{1}{2} z(1 + r^2 + \frac{1}{3} z^2)$$

transforms ω to the standard form.

So the contact structure, defined by $\omega = 0$, is equivalent to the standard structure in R^3.

CHAPTER 1

GEOMETRY OF TRANSVERSAL FIELDS

All contact forms on a manifold M are equivalent locally, i.e.

$$f^*\omega_1 = \omega_0$$

is satisfied for a diffeomorphism f, defined in a neighbourhood of a point. This is no longer true for transformations which are not local, however, not even when the forms define identical contact structures. We may have to multiply one of the forms by a suitable function, non-zero everywhere. One obvious reason is that the volume of M with respect to $\omega \wedge (d\omega)^n$ is invariant.

In §1 we find another, more geometric reason. The so called transversal fields of the two forms must be equivalent. We prove this and see how it works in a few examples.

In §2 we find the transversal field of the non-standard form $r\sin r d\vartheta + \cos r dz$, introduced on p. 10.

In §3 we show that all contact structures of the type $r^2 f(r) d\vartheta + g(r) dz = 0$, where $f(r) = g(r) = 1$ near $r = 0$, are equivalent to the standard structure if $f(r) \neq 0$. These structures have cylinder symmetry, and the contact planes rotate less than 180° from the z-axis.

In §4 we discuss the standard contact form on S^3, which we introduced on p. 17, and we study its stereographic image in R^3.

We assume all functions and forms are sufficiently regular to give transformations which are continuously differentiable twice.

§1. The transversal field of a contact form

Let ω be a contact form on an orientable, $(2n+1)$-dimensional, differentiable manifold M. There is a unique vector field V such that

(i) $\omega(V) = 1$

(ii) the Lie derivative $L_V\omega = 0$.

(See for example [4, p. 135])

The vector field V will be called the transversal field of ω, as V is transversal to the contact planes at each point of M. The trajectories of particles moving along the transversal field will be called the transversal curves.

REMARK The vector field V generates a 1-parameter group of diffeomorphisms φ_t, which may be a global group. The condition $L_V\omega = 0$ means that $\varphi_t^*\omega = \omega$, i.e. the flow of V preserves ω.

Let V and W be two vector fields, which generate the 1-parameter groups φ_t and ψ_t, resp. We say that V is equivalent to W if there is a diffeomorphism f of M, such that

$$\varphi_t \circ f = f \circ \psi_t.$$

THEOREM Let ω_0 and ω_1 be two equivalent contact forms on an orientable, $(2n+1)$-dimensional, differentiable manifold M, i.e. there is a diffeomorphism f of M, such that

$$f^*\omega_1 = \omega_0.$$

Then the two forms have equivalent transversal fields.

Proof: Let V_0 be the transversal field of ω_0 and V_1 the transversal field of ω_1, and let the corresponding 1-parameter groups be φ_t and ψ_t, resp. As

$$(f^{-1} \circ \varphi_t \circ f)^* \omega_1 = \omega_1,$$

it follows that the vector field E, which generates $f^{-1} \circ \varphi_t \circ f$, satisfies $L_E \omega_1 = 0$.

We have $\omega_1(E) = 1$ as $\omega_0(V_0) = 1$. As the transversal field is unique, it follows that $E = V_1$, so $f^{-1} \circ \varphi_t \circ f = \psi_t$, i.e. V_0 and V_1 are equivalent. Q.E.D.

As vector fields are not always equivalent, this theorem explains why it is necessary in general to multiply one of the forms by a function, non-zero everywhere, in order to obtain equivalence, even when the associated volumes are the same.

We shall now consider some explicit examples in R^3. Let $Xdx + Ydy + Zdz$ be a contact form. The transversal field of this form is

$$\frac{1}{X(Y_z - Z_y) + Y(Z_x - X_z) + Z(X_y - Y_x)} (Y_z - Z_y, Z_x - X_z, X_y - Y_x).$$

REMARK If we let $\underline{A}$ be the vector field (X,Y,Z), then the transversal field is $(\underline{A} \cdot \text{curl } \underline{A})^{-1} (\text{curl } \underline{A})$.

We shall now find the transversal fields and the transversal curves of the contact forms $xdy + dz$, $\sin x dy + \cos x dz$ and $xdy - ydx + dz$, which we described in the Introduction, §2.

EXAMPLE 1 The form $xdy + dz$ has transversal field $(0,0,1)$, i.e. the vectors are of unit length and parallel to the z-axis. The transversal curves are $x = \text{constant}$, $y = \text{constant}$, $z = t$.

EXAMPLE 2 The transversal field of $\sin x\,dy + \cos x\,dz$ is $(0,\sin x,\cos x)$, i.e. the field vectors are orthogonal to the contact planes. The transversal curves are $x = \text{constant}$, $y = t\cdot\sin x$, $z = t\cdot\cos x$. (Fig. 10)

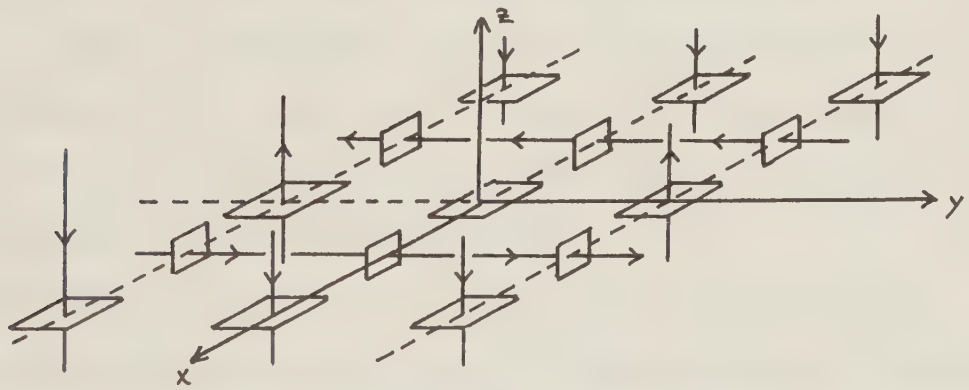


Fig. 10

EXAMPLE 3 The form $xdy - ydx + dz$ has transversal field $(0,0,1)$, i.e. the same field as the form in example 1.

We shall next see how we can use information about the transversal field to study equivalence of contact forms in R^3.

EXAMPLE 1 The form $xdy + h(z)dz$ is a contact form if $h(z) \neq 0$. In this example we show that the contact structure $xdy + h(z)dz = 0$ is equivalent to the standard structure.

The transversal field of $xdy + h(z)dz$ is $(0,0,1/h(z))$, so the transversal curves are $x = \text{constant}$, $y = \text{constant}$, $t = \int_0^z h(s)ds$. If $\int_{-\infty}^{\infty} |h(s)|ds < \infty$, then we cannot map these curves point for point to $x = \text{constant}$, $y = \text{constant}$, $z = t$, i.e. the transversal fields are not equivalent, and therefore the form $xdy + h(z)dz$ is not equivalent to $xdy + dz$.

We shall now multiply one of the forms by a suitable function $\lambda \neq 0$, after which the forms will be equivalent. As $h(z) \neq 0$, we can multiply the form $xdy + h(z)dz$ by $1/h(z)$, and obtain the form $(x/h(z))dy + dz$.
The transversal field of this form is $((x \cdot h'(z))/h(z),0,1)$, and the transversal curves are $x = \text{constant} \cdot h(t)$, $y = \text{constant}$, $z = t$. These curves can be mapped to $x = \text{constant}$, $y = \text{constant}$, $z = t$.

We now find that the map $\phi = x/h(z)$, $\varphi = y$, $\psi = z$ is a diffeomorphism of R^3, such that

$$(x/h(z))dy + dz = \phi\, d\varphi + d\psi.$$

We can also multiply $\phi\, d\varphi + d\psi$ by $h(\psi)$, and obtain

$$xdy + h(z)dz = h(\psi)(\phi\, d\varphi + d\psi).$$

It follows that the contact structure $xdy + h(z)dz = 0$ is equivalent to the standard structure.

EXAMPLE 2 We show in this example, that the transversal field of a contact form can contain closed trajectories, and therefore cannot be equivalent to (0,0,1).

Consider the contact form

$$e^{-r^2}(r^2 d\vartheta + dz).$$

The transversal field is

$$dr/dt = 0, \qquad d\vartheta/dt = e^{r^2}, \qquad dz/dt = e^{r^2}(1 - r^2).$$

The transversal curves are

$$r = r_0, \qquad \vartheta = t \cdot e^{r_0^2}, \qquad z = t \cdot (1 - r_0^2) \cdot e^{r_0^2}.$$

The transversal curves are circles for $r = r_0 = 1$. (Fig. 11)

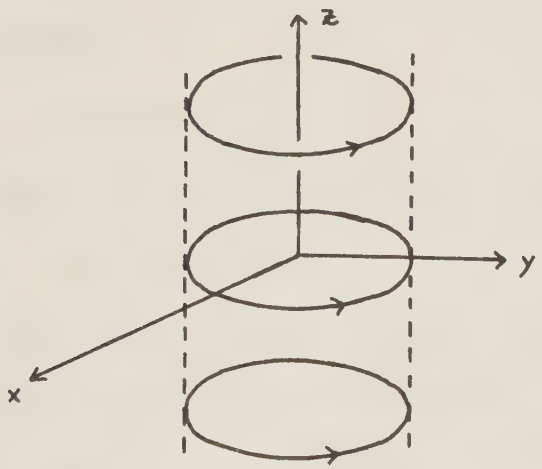


Fig. 11

It follows that the transversal field of $e^{-r^2}(r^2 d\vartheta + dz)$ cannot be equivalent to (0,0,1), so the forms $r^2 d\vartheta + dz$ and $e^{-r^2}(r^2 d\vartheta + dz)$ cannot be equivalent. The forms determine identical contact structures, however.

EXAMPLE 3 The form $g(x)dy + dz$ is a contact form if $g'(x) \neq 0$. In this example we show that this form is equivalent to the standard form $\Phi d\Psi + d\Psi$.

A transformation to the standard form is for example given by $\Phi = g(x)$, $\Psi = y$, $\Psi = z$, which is a one-to-one map, as $g(x)$ is strictly monotone. This map is not onto, however, if $g(x)$ is a bounded function. We shall find another map, which is both one-to-one and onto for all $g(x)$ such that $g'(x) \neq 0$.

The transversal field of the form $g(x)dy + dz$ is $(0,0,1)$. From this follows that Φ and Ψ are functions of x and y only. We try to determine a map of the following type

$$\Phi = y \cdot H(x), \qquad \Psi = x, \qquad \Psi = y \cdot K(x) + z.$$

We find the unique solution $H(x) = -g'(x)$, $K(x) = g(x)$. The transformation

$$\Phi = -y \cdot g'(x), \quad \Psi = x, \qquad \Psi = y \cdot g(x) + z \qquad (1)$$

is a diffeomorphism of R^3, so the contact form $g(x)dy + dz$ is equivalent to the standard form.

EXAMPLE 4 The form $\omega = f(x)dx + g(x)dy + h(x)dz$ is a contact form if $g'(x) \cdot h(x) - g(x) \cdot h'(x) \neq 0$. This form is also equivalent to the standard form $\Phi d\Psi + d\Psi$. By using the transformation (1) in example 3 we find that

$$\Phi = -y \cdot g'(x) - z \cdot h'(x), \qquad \Psi = x, \qquad \Psi = y \cdot g(x) + z \cdot h(x) + \int_0^x f(s)ds$$

transforms ω to the standard form. We can solve uniquely for x, and as $g'(x) \cdot h(x) - g(x) \cdot h'(x) \neq 0$, we can also solve uniquely for y and z, so we have a diffeomorphism of R^3. (Compare this example to example 4 in the Introduction, §3).

§2. The transversal field of the non-standard form

We introduced the non-standard form $r\sin r\,d\vartheta + \cos r\,dz$ on p. 10. If we let $J = 1 + \sin 2r/2r$, then the transversal field is

$$dr/dt = 0, \qquad J\cdot(d\vartheta/dt) = \sin r/r, \qquad J\cdot(dz/dt) = \sin r/r + \cos r.$$

The transversal curves are circles at $r = r_1 < r_2 < \ldots$, where r_i are the successive roots of $\sin r/r + \cos r = 0$. (Fig. 12)

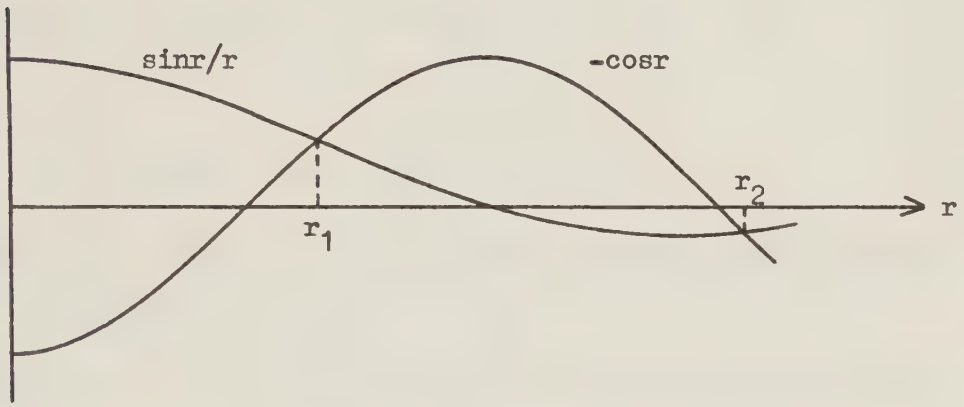


Fig. 12

It follows that the form $r\sin r\,d\vartheta + \cos r\,dz$ is not equivalent to the form $r^2 d\vartheta + dz$, which has transversal field $(0,0,1)$.

In this paper we show by geometric arguments, that the contact structure $r\sin r\,d\vartheta + \cos r\,dz = 0$ is not equivalent to the standard structure. It follows that there is no function λ, non-zero everywhere, such that

$$r\sin r\,d\vartheta + \cos r\,dz = \lambda\,(R^2 d\theta + d\Psi),$$

where R, θ, Ψ is a diffeomorphism.

§3. An equivalence theorem

In this paper we show, that for example the structure $r\sin r d\vartheta + \cos r dz = 0$ is not equivalent to the standard structure. The non-equivalence depends on the radial symmetry with respect to the z-axis, and on the fact that the contact planes rotate at least 180° from the z-axis.

In this section we prove the following equivalence theorem.

THEOREM Let $f(r) = g(r) = 1$ near $r = 0$ and assume that $f(r) \neq 0$ everywhere. All contact structures of the type

$$r^2 f(r) d\vartheta + g(r) dz = 0$$

are then equivalent to the standard structure.

REMARK 1 The condition $f(r) \neq 0$ means that the contact planes rotate less than 180° from the z-axis.

REMARK 2 It is no restriction from the geometric point of view to assume that $f(r) = g(r) = 1$ near $r = 0$.

The theorem follows from the following result.

LEMMA Let $h(r) = 1$ near $r = 0$. The contact form $r^2 d\vartheta + h(r) dz$ is then equivalent to the standard form $R^2 d\theta + d\Psi$.

Proof of the lemma: Let $J = rh'(r) - 2h(r)$, which is non-zero, as we have a contact form. The transversal field of the form is

$$dr/dt = 0, \qquad J \cdot (d\vartheta/dt) = h'(r)/r, \qquad J \cdot (dz/dt) = -2.$$

The transversal curves are the spirals

$$r = R, \qquad \vartheta = -(h'(r) \cdot z)/2r, \qquad z = -2t/J.$$

The transformation

$$R = r, \qquad \theta = \upsilon + (h'(r)\cdot z)/2r, \qquad \Psi = -(J\cdot z)/2$$

therefore maps the transversal field of $r^2 d\upsilon + h(r)dz$ to $(0,0,1)$. The map R, θ, Ψ, is a one-to-one transformation of space onto itself, which also transforms $r^2 d\upsilon + h(r)dz$ to the standard form. The transformation is a diffeomorphism as we assumed that $h(r) = 1$ near $r = 0$. It follows that the form $r^2 d\upsilon + h(r)dz$ is equivalent to the standard form. Q.E.D.

REMARK It is clear from the proof that we would also get a contact diffeomorphism if we assume that $h'(r)/r$ is continuously differentiable twice. For regularity conditions we must also assume that $h'(0) = 0$.

Proof of the theorem: As $f(r) \neq 0$, we can consider the contact form $r^2 d\upsilon + (g(r)/f(r))dz$. From the lemma follows that this form is equivalent to the standard form $R^2 d\theta + d\Psi$. If we multiply both forms by f, we get

$$r^2 f(r) d\upsilon + g(r)dz = f\cdot(R^2 d\theta + d\Psi). \qquad \text{Q.E.D.}$$

§4. The standard contact structure on S^3

In the Introduction, §7, we found that the form

$$x_1dx_2 - x_2dx_1 + x_3dx_4 - x_4dx_3 \tag{1}$$

defines a contact structure on S^3, which we called the standard structure. In this section we shall discuss this structure and its stereographic image in R^3.

The transversal field of (1) is given by

$$dx_1/dt = -x_2, \quad dx_2/dt = x_1, \quad dx_3/dt = -x_4, \quad dx_4/dt = x_3.$$

The corresponding curves are the so called Hopf fibres, which are circles in S^3, mutually linked [5]. See also [22, p. 300]. We see that the transversal field of (1) is orthogonal to the contact planes in S^3.

The form (1) is transformed under stereographic projection to the contact form $\lambda\omega$, where

$$\lambda = \frac{4}{(1 + x^2 + y^2 + z^2)^2},$$

$$\omega = (xz - y)dx + (yz + x)dy + \tfrac{1}{2}(1 + z^2 - x^2 - y^2)dz,$$

as we stated on p. 19. In cylinder coordinates we have

$$\omega = zrdr + r^2d\vartheta + \tfrac{1}{2}(1 + z^2 - r^2)dz.$$

The transversal curves in S^3 of the differential form (1), i.e. the Hopf fibres, are mapped to circles in R^3 under stereographic projection, except for the circle through the north pole, which is mapped to the z-axis. The form $\lambda\omega$ has therefore circles among the transversal curves, and cannot be equivalent to the standard form.

We shall now consider the contact form ω. The structure $\omega = 0$ has radial symmetry with respect to the z-axis, and the contact planes rotate more than 90° from the z-axis.

The transversal field of ω is parallel to the field

$$dr/dt = 0, \qquad d\vartheta/dt = 1, \qquad dz/dt = 1.$$

The corresponding curves are

$$r = R, \qquad \vartheta = z, \qquad z = t.$$

As the transversal curves are spirals with $dz/dt \neq 0$ in cylinders parallel to the z-axis, it follows that the contact planes rotate less than 180° from the z-axis.

We shall now determine the diffeomorphism we stated on p. 19, and which transforms ω to the standard form $R^2 d\theta + d\psi$.

Let $J = 1 + r^2 + z^2$. The transversal field of ω is then

$$dr/dt = 0, \qquad J\cdot(d\vartheta/dt) = 2, \qquad J\cdot(dz/dt) = 2.$$

It follows that the transformation

$$R = r, \qquad \theta = \vartheta - z, \qquad \psi = \frac{1}{2} z(1 + r^2 + \frac{1}{3} z^2)$$

maps the transversal field of ω to $(0,0,1)$. The map also transforms $R^2 d\theta + d\psi$ to the form ω. The polynomial $z(1 + r^2 + \frac{1}{3} z^2)$ is a strictly increasing function of z, so we obtain a diffeomorphism of R^3.

CHAPTER 2

GEOMETRY OF INTEGRAL CURVES

Integral curves were introduced by Lie. We defined and discussed them in the Introduction, §5. In this chapter we study integral curves in the standard and non-standard contact structures.

In §1 we prove Lie's non-equivalence theorem, which we stated in the Introduction, §3.

In §2 we find that the problem, whether the standard and non-standard contact structures are equivalent or not, depends on the geometry of flat bands.

In §3 we introduce a basic concept, the kinking number of a projection of a closed curve in space.

In §4 we study geometric conditions on integral curves.

In §5 we study the geometry of integral curves as a geometry of loops.

In §6 we observe that for integral curves in the standard structure $xdy + dz = 0$, there is always a projection in which the crossings twist as a right-handed screw.

In §7 we formulate two versions of a knotting criterion, which we shall prove in Chapter 3. It follows from this criterion and the geometry of flat bands, that the standard and non-standard contact structures are non-equivalent.

In §8 we give an example which shows that we cannot use the Alexander polynomial to prove our knotting criterion. We must instead develop a completely geometric proof.

§1. Lie's non-equivalence theorem

We stated Lie's non-equivalence theorem on p. 9. He proved that the contact structure $xdy - ydx + dz = 0$ is not equivalent under projective transformations to the structure $xdy + dz = 0$. We give in this section a somewhat different proof, which can serve as an introduction to our discussion later.

The structure $xdy - ydx + dz = 0$ is defined in cylinder coordinates by $r^2 d\vartheta + dz = 0$. It follows that the lines ϑ = constant, z = constant are integral lines. The lines for z = 0 are illustrated in figure 13.

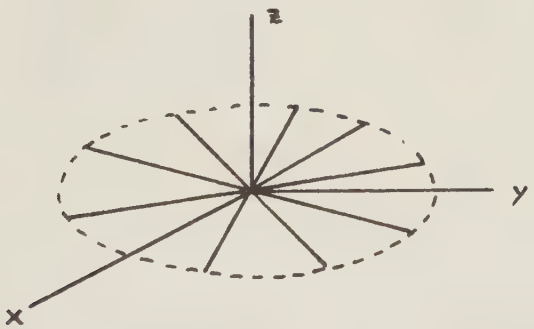


Fig. 13

Under a projective transformation, the plane z = 0 is mapped to a plane, and the integral lines in fig. 13 are mapped to integral lines intersecting in one point in the image plane.

Now consider the structure $xdy + dz = 0$. It is easy to verify that the only planes which intersect this structure along lines are $z = \text{constant}$ and $x = \text{constant}$.

The integral lines in the first case are all lines for which $dy = dz = 0$, i.e. of all lines orthogonal to the yz-plane. The integral lines in the second case are those lines for which $x = -dz/dy$, and which are illustrated in figure 14.

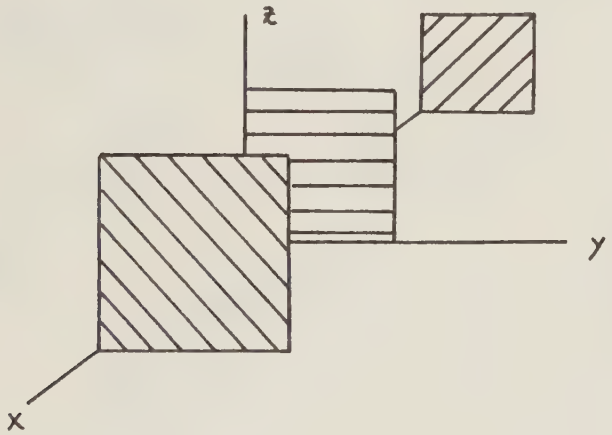


Fig. 14

It follows, that there is no plane, which intersects the structure $xdy + dz = 0$ along integral lines which meet in one point. This completes the proof of Lie's theorem.

§2. Flat bands

Consider a non-standard structure, for example $r\sin r d\vartheta + \cos r dz = 0$, and the disc $\{0 \leq r \leq \pi,\ z = 0\}$. The boundary circle $\{r = \pi,\ z = 0\}$ is an integral curve, and the radii $\{0 \leq r \leq \pi,\ \vartheta = \text{constant},\ z = 0\}$ are integral lines. (Fig. 15)

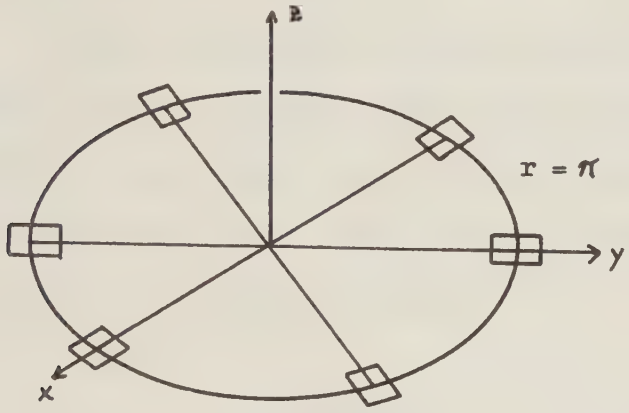


Fig. 15

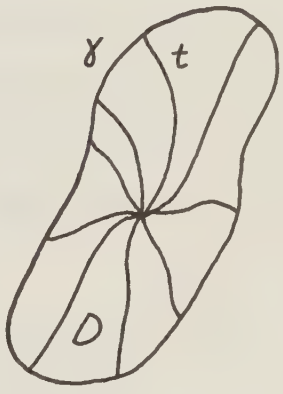


Fig. 16

Now assume the non-standard structure were equivalent to the standard structure $xdy + dz = 0$. Then the disc above would be mapped to a disc D embedded in space in the following way. The boundary γ of the disc would be an integral curve of the standard structure. The disc would also intersect the contact planes in a family of integral curves t, which would be transversal to the boundary at each point. (Fig. 16)

So in order to prove that the two structures are not equivalent, we have to prove that we cannot construct a disc with the properties described above. The following consideration gives a clue to what ideas we need for such a proof.

Assume the two structures were equivalent. It would follow that a band $\{\pi - \varepsilon \leq r \leq \pi\ ,\ z = 0\}$ along the integral circle $\{r = \pi\ ,\ z = 0\}$ in the non-standard structure, would be mapped to a band N along a closed integral curve γ in the standard structure $xdy + dz = 0$. The integral lines $\{\pi - \varepsilon \leq r \leq \pi\ ,$ ϑ = constant, $z = 0\}$ would be mapped to integral curves t in the band N, transversal to γ.

We have illustrated in fig. 17 the integral curve γ, the band N, and the integral curves t transversal to γ. We have given γ figure-eight shape, because we shall see in §4 that a closed integral curve in the standard structure $xdy + dz = 0$ has at least one over-crossing.

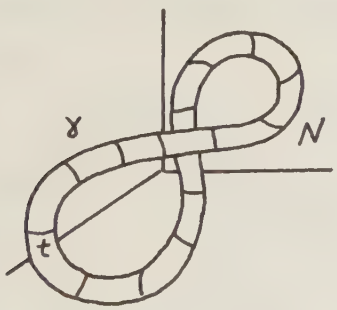


Fig. 17

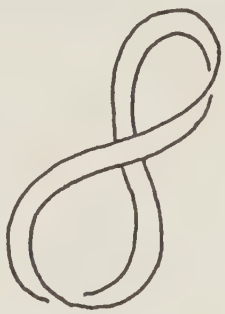


Fig. 18

The band N is tangent to the contact planes along γ, because of the integral curves t. So the band does not twist. We therefore call it a flat band. Compare this to a rubber band in figure-eight position. The band then twists 360°. (Fig. 18)

We shall next observe that it is not possible to extend the band in fig. 17 to a disc. This is so, because the boundaries of the band are linked. To see this, we shall first briefly discuss linking of curves. (For details, see [19, p. 132ff])

Two closed curves in space are <u>linked</u> if they cannot be separated. (Fig. 19)

Fig. 19

A pair of closed curves in space can be assigned an integer, the so called <u>linking number</u>, l. If $l \neq 0$, then the two curves are linked. We shall now explain how to determine the linking number.

Let γ_1 and γ_2 be two closed curves. Each curve is supposed to be oriented, and all crossings are assumed to be separated. (Fig. 20)

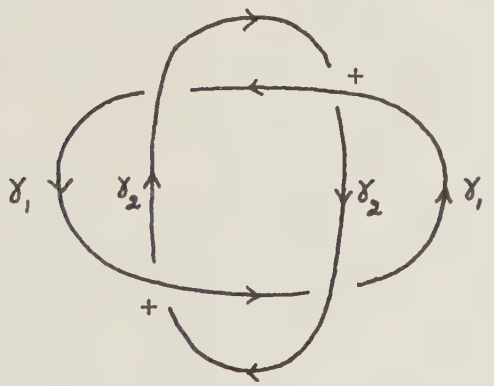


Fig. 20

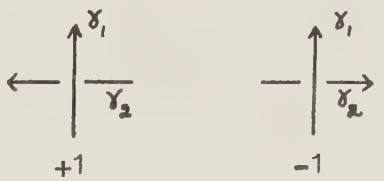


Fig. 21

Now assign +1 to each crossing where one of the curves, say γ_2, passes <u>under</u> γ_1 to the <u>left</u>, and -1 to each crossing where γ_2 passes <u>under</u> γ_1 to the <u>right</u>. (Fig. 21) The sum of these numbers is the linking number of the two curves. The linking number of the two curves in figure 20 is +2.

Two unlinked curves have linking number l = 0. However, two curves can be linked and yet have linking number 0. Figure 22 illustrates such a case.

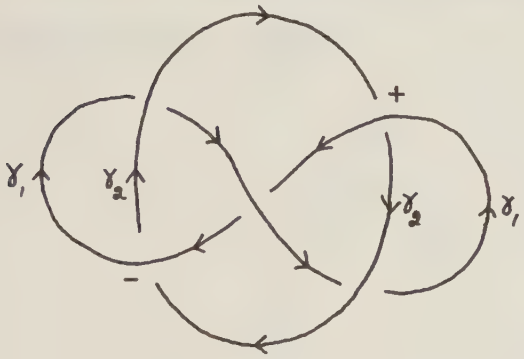


Fig. 22

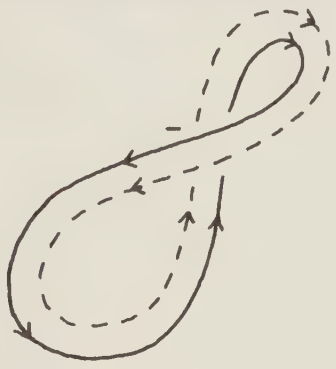


Fig. 23

Now we can see that the boundaries of the flat band in fig. 17 on p. 36 are linked. If we assign the same orientation to each boundary, we find that their linking number is -1. (Fig. 23)

As the two boundary curves in fig. 23 are linked, they cannot be curves on a disc. We look for a disc, whose boundary γ is an integral curve of the standard structure $xdy + dz = 0$, and has integral curves t transversal to γ. From the discussion above follows that we must first find a flat band, whose boundaries are unlinked. This can only be possible if their linking number is 0.

In order to proceed further, it is now convenient to first introduce a new concept, the kinking number of a regular projection of a closed curve in space. We describe this in the next section.

§3. The kinking number

Consider a regular projection of a closed curve γ in space, i.e. we assume we look at the curve from a certain direction, and we assume all crossings are separated. We also give the curve an orientation. (Fig. 24)

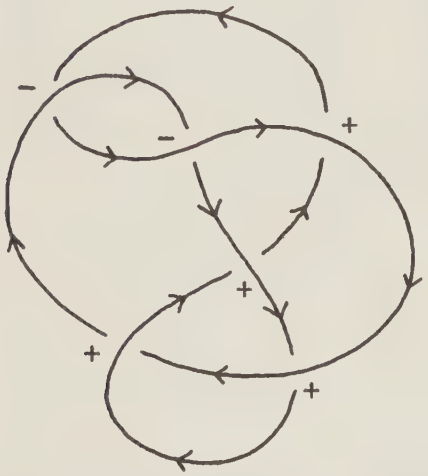


Fig. 24

Now assign +1 to each crossing where the curve passes under itself to the left, and -1 to the other crossings. The sum of these numbers is the kinking number $\mathcal{k}$ of the projection of the curve. The kinking number is independent of the orientation of the curve. The curve in fig. 24 has kinking number $\mathcal{k} = +4 - 2 = +2$.

We shall refer to this definition of $\mathcal{k}$ as the arithmetic definition of the kinking number. In Chapter 3, §5, we shall give a geometric description of the kinking number of a closed curve γ, as the number of times an orientable surface with boundary γ twists around γ.

Consider now a flat band, and assume the boundary curves have the same orientation. (Fig. 25)

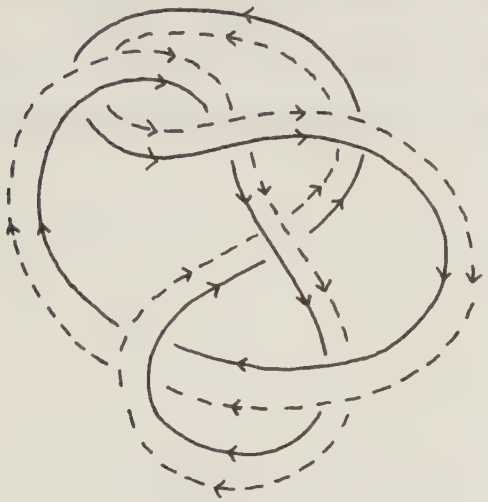


Fig. 25

As the band does not twist, it follows that the boundary curves have the same kinking number. We also see that this kinking number equals the linking number of the two boundary curves.

Now consider a flat band, one of whose boundary curves γ is an integral curve of the standard structure $xdy + dz = 0$. In order to construct an embedded disc with boundary γ, and which contains the integral curves t transversal to γ, we must first look for a band whose boundary curves have linking number 0. As this linking number equals the kinking number of γ, it follows that it is enough to look for an integral curve whose kinking number is 0. However, we shall show in Chapter 3, that an integral curve of $xdy + dz = 0$ with kinking number 0, cannot bound an embedded disc. It will follow that the standard and non-standard contact structures are not equivalent.

We observe that the kinking number of an integral curve should be calculated with respect to a projection, where we look at the curve from above or from below, along the z-axis. We choose to look at the curves from above, and we call this projection the kinking number projection. (Fig. 26)

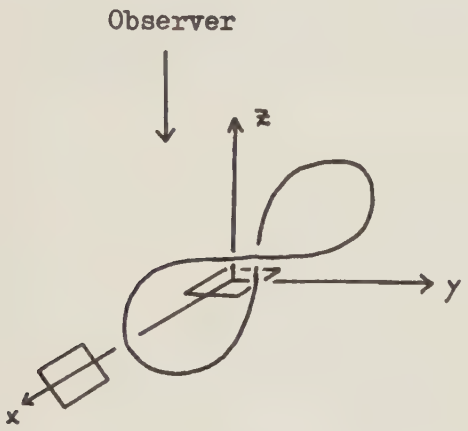


Fig. 26

The kinking number in this projection will be denoted by k. We shall next find out how to study integral curves in this projection.

§4. Loops in the kinking number projection

We consider the standard structure $xdy + dz = 0$, and we shall study its closed integral curves. In this section we look at the curves from above along the z-axis, i.e. we consider the kinking number projection.

Our first observation is that a closed integral curve must have at least one over-crossing. This follows from the following consideration.

Let $\gamma(t) = (x(t),y(t),z(t))$ be a closed integral curve, so $\gamma(0) = \gamma(1)$, and $dz = -xdy$. The projection of the curve in the xy-plane has at least one point of self-intersection as γ is closed. Assume the projection intersects itself for $t = t_0$ and $t = t_1$, $t_0 < t_1$. Figure 27 illustrates the two possible cases.

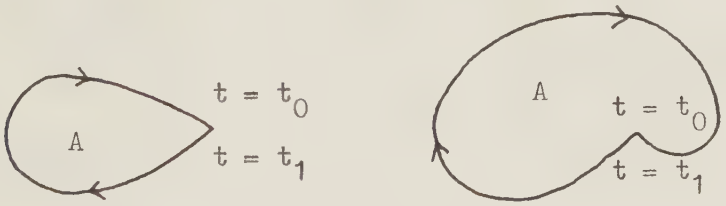


Fig. 27

Let A be the area enclosed by the projection of γ in the xy-plane for $t_0 \leq t \leq t_1$. From Stokes' theorem now follows,

$$z(t_1) - z(t_0) = \int_{t_0}^{t_1} dz = \int_{t_0}^{t_1} -xdy = -\iint_A dxdy > 0.$$

As $z(t_1) > z(t_0)$ it follows that $\gamma(t_1)$ is above $\gamma(t_0)$, i.e. the curve has at least one over-crossing.

The part of γ in space between t_0 and t_1 is called a loop. We have found that a loop in the standard structure $xdy + dz = 0$ always looks as in fig. 28 or as in fig. 29. The kinking number in the first case is -1, and in the second case +1.

Fig. 28

Fig. 29

We shall now consider a configuration of curve segments, where each segment passes under the next segment at each crossing, when we go around the configuration clockwise. (Fig. 30)

Fig. 30

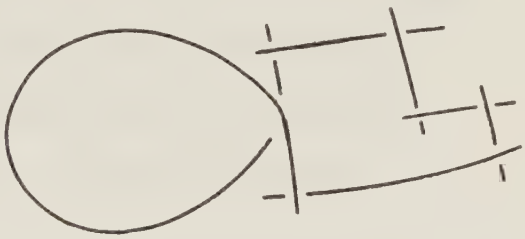


Fig. 31

Such a configuration cannot be part of an integral curve. We have to insert a loop in one of the curve segments. (Fig. 31) We call such a loop a correction loop. A correction loop has always kinking number -1.

That we need a correction loop can be seen as follows. Join the segments by integral curves, except at one crossing C. It then follows from Stokes' theorem that the curve must pass over itself at C, instead of under. So we must insert a correction loop in order to obtain an integral curve.

§5. Geometry of loops

Assume that a projection of a closed curve is given, and that the curve is not necessarily an integral curve in the structure $xdy + dz = 0$. We shall describe a method to modify such a projection into an integral curve in the kinking number projection, without changing the knot type. We do this modification by inserting correction loops. From this point of view, the geometry of integral curves can be said to be a geometry of loops.

Note that each correction loop contributes -1 to the kinking number. The method we shall describe is very useful in order to estimate what kinking number we can maximally obtain for a projection we take from a knot table, for example. We will assume that a correction loop has no crossings with the original curve, except for the crossing of the loop with itself.

We shall describe the procedure in a specific example. Consider the projection in figure 32. It is knot 5_2 in [19, p.391]. It is clear that we need at least the four correction loops indicated in figure 33 to obtain an integral curve.

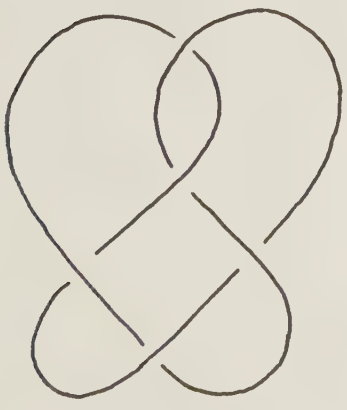


Fig. 32

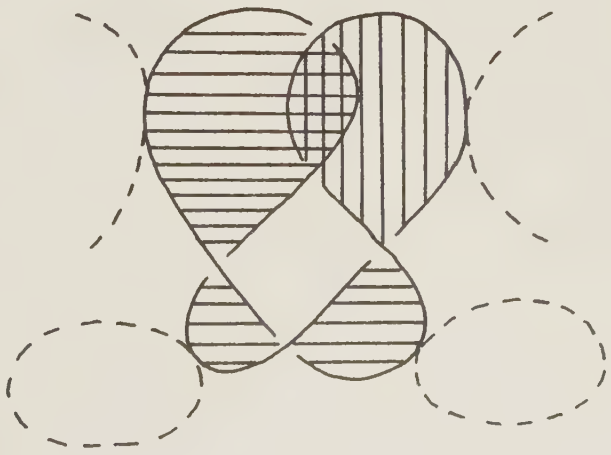


Fig. 33

We now give the modified knot an orientation, and we find that the kinking number is +1. (Fig. 34)

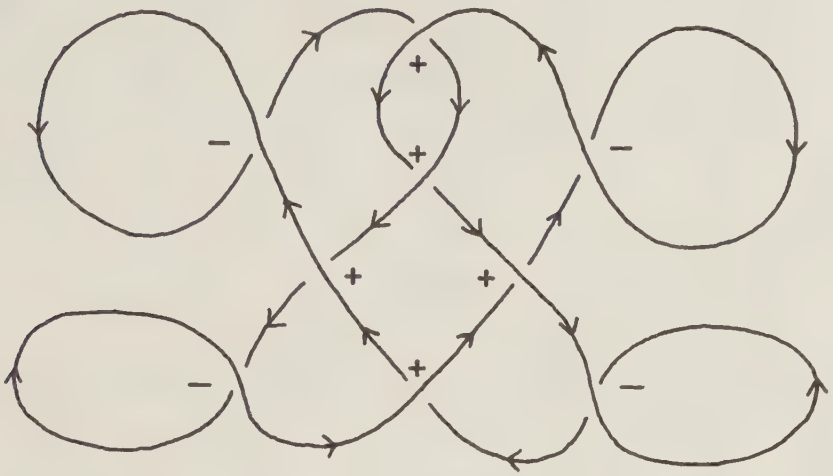


Fig. 34

In order to determine a minimal number of correction loops, i.e. maximal possible kinking number, we proceed as follows. Assign a colour to each configuration which needs a loop, i.e. draw each curve segment in such a configuration with the same colour. Then place the correction loops in the coloured segments in such a way that we need as few loops as possible.

The kinking number of the knot in fig. 34 is +1. The colour method shows that we cannot do with fewer than four loops, i.e. we can say that $k \leq 1$ when the original projection is embedded as an integral curve. It is in this way we can estimate k. We shall next prove that the loops we have indicated in fig. 34 are sufficient.

We give each crossing a number, except for the loop crossings. We label the bounded regions, including the loop regions, $r_1, r_2, \ldots$, where r_i will also be the area of the region. (Fig. 35)

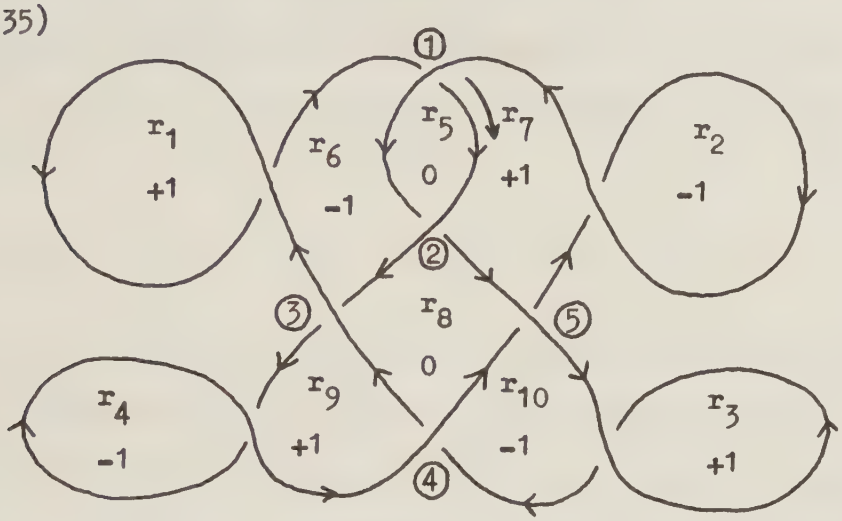


Fig. 35

We have given the knot an orientation, and we now assign to each region its <u>winding number</u> w_i, which we shall now explain. The winding number of a region is the net rotation of the curve around the region. We adopt the conventions that counterclockwise rotation counts as positive winding, and that the unbounded region has winding number 0. The winding number of a region can be found in the following way. Draw a line from the region, which intersects the curve, but not at a crossing. The winding number is then the number of times the curve intersects the line from right to left, minus the number of times the curve intersects the line from left to right.

We have given the winding numbers for each region in figure 35.

We want the projection in fig. 35 to be an integral curve. Then the areas r_i cannot be arbitrary. They must be compatible with the crossings, and with the fact that the curve is closed. From Stokes' theorem follows that the curve is closed if and only if $\sum w_i r_i = 0$. In our example this closed curve condition is (Fig. 35)

$$r_1 - r_2 + r_3 - r_4 \qquad - r_6 + r_7 \qquad + r_9 - r_{10} = 0. \tag{C}$$

Now consider the crossings, ① say. (Fig. 35) We go along the curve, starting from ① , in the direction of the arrow, for example, and we draw the projection we follow until we come back to ① . (Fig. 36) We now consider this curve as a new projection, and we mark the areas of the regions, and assign new winding numbers.

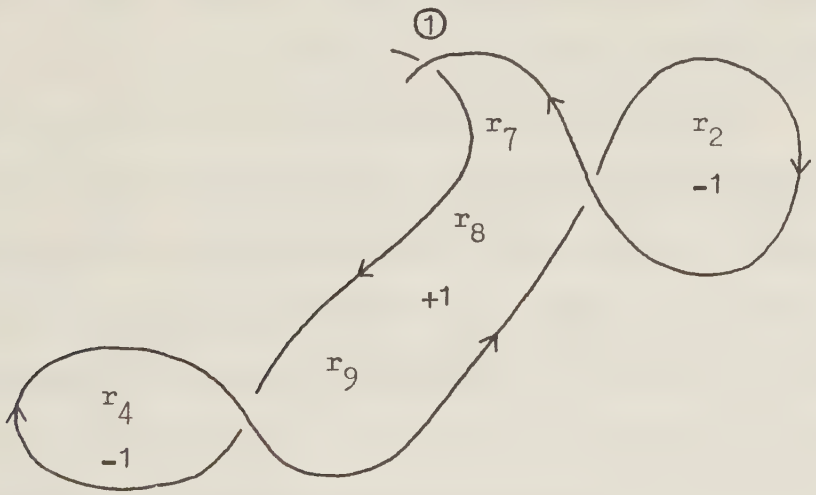


Fig. 36

The condition that the curve passes over itself in ① can be expressed in terms of an inequality. We get

$$- r_2 - r_4 + r_7 + r_8 + r_9 < 0. \tag{1}$$

For the other crossings we get the inequalities (the number of the inequality refers to the crossing with that number in fig. 35)

$$-r_2 - r_4 + r_5 + r_7 + r_8 + r_9 > 0, \tag{2}$$

$$r_1 - r_5 - r_6 > 0, \tag{3}$$

$$r_1 - r_4 - r_5 - r_6 + r_9 < 0, \tag{4}$$

$$-r_2 + r_5 + r_7 < 0. \tag{5}$$

We solve the inequalities (1) - (5), together with the closed curve condition (C). The following table gives a solution.

r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}
10	10	10	10	4	5	5	8	5	5

We can now embed the knot as an integral curve in the kinking number projection. It follows that the maximal possible kinking number for this projection with correction loops is +1.

The system of inequalities can often be quickly solved when the knot projection has a symmetry, and we look for a symmetric solution. If r_i and r_j are symmetric regions, we can let $r_i = r_j$, which reduces the number of unknowns. In our case above, we looked for a solution such that $r_1 = r_2 = r_3 = r_4$ and $r_6 = r_7 = r_9 = r_{10}$. The condition (C) is then automatically satisfied, and it is easy to find a solution of the inequalities (1) - (5).

In the general case, however, the system of inequalities becomes unmanageable when the number of crossings increases. We shall next describe another method, with which we can immediately decide whether a curve is an integral curve or not.

§6. The positive twist projection

Consider the standard contact structure $xdy + dz = 0$. An integral curve of this structure is a curve which at each point of space satisfies $x = -dz/dy$. This relation can be interpreted in the following way.

Assume we observe an integral curve along the x-axis, for example from $+\infty$, i.e. we are the observer in figure 37.

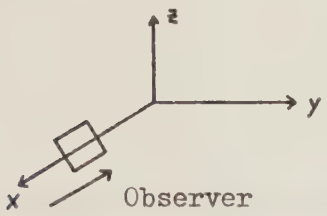


Fig. 37

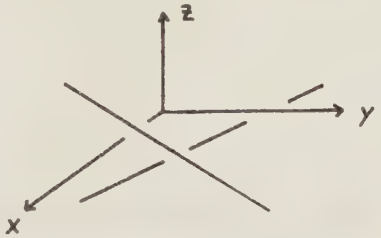


Fig. 38

It then follows from the condition $x = -dz/dy$, that a crossing always twists as a right-handed screw in the y-direction. (Fig. 38) Such a crossing is said to have a positive twist, and we therefore call this projection the positive twist projection. This projection has a spike at points where the curve changes y-direction. Such a spike is an example of a Legendre singularity [1, p. 367]. We have illustrated an integral curve in the positive twist projection in figure 39, and the corresponding curve in the kinking number projection in figure 40.

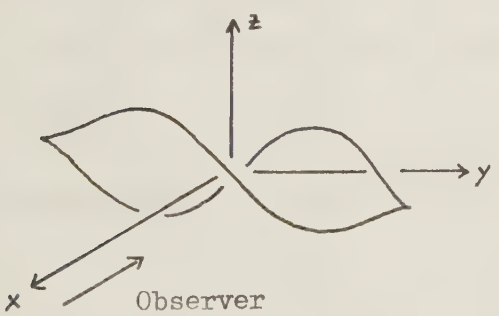


Fig. 39

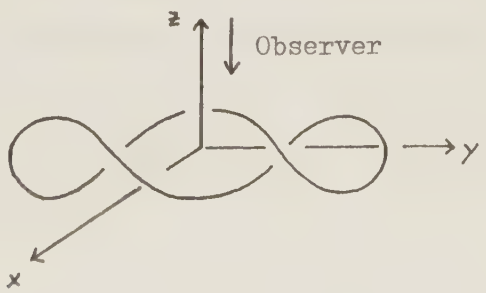


Fig. 40

We observe that dx/dy changes sign where the integral curve changes y-direction. From the relation $dx/dy = -d^2z/dy^2$ therefore follows, that d^2z/dy^2 changes sign at a spike. This means that the parts of the positive twist projection in a neighbourhood above and below a spike have different directions of concavity. (Fig. 41)

Fig. 41

Consider a closed integral curve of $xdy + dz = 0$ with b spikes to the right (and to the left) in the positive twist projection. Let $k(\alpha)$ be the kinking number in a projection in the direction of the x-axis along lines which intersect the contact planes under an angle α, $0 \leq \alpha \leq 90^\circ$. (Fig. 42)

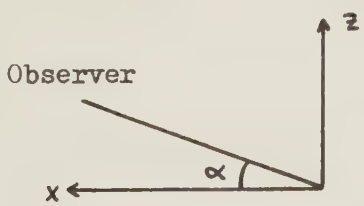


Fig. 42

The kinking number of a curve depends in general on the projection of the curve. In our case, however, we have the following important result.

Proposition $\quad k(\alpha) = k(0) - b, \quad 0 < \alpha \leq 90^\circ.$

As $k(0)$ is the kinking number in the positive twist projection, and $k(90^{\circ})$ is the kinking number in the kinking number projection, we have immediately the following fundamental rule.

$$\boxed{k = \mathcal{k} - b}$$

k = kinking number in the kinking number projection,

$\mathcal{k}$ = kinking number in the positive twist projection,

b = number of spikes pointing to the right (or to the left).

Proof of the proposition: We shall first describe the projection for small α. The crossings are the same as for $\alpha = 0$, but the spikes disappear. Each spike to the left becomes a smooth curve, and each spike to the right becomes a small loop. The curve in figure 39 for example, will look as in figure 43.

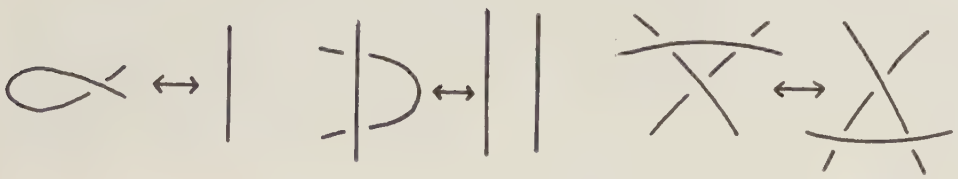
Fig. 43

As a loop which appears at a spike has kinking number -1, we have found that

$$k(\alpha) = k(0) - b, \quad \alpha \gtrsim 0. \tag{1}$$

The curve we see for small α can look very different from the curve we see for larger values of α. When α increases, overcrossings appear and disappear at a finite number of values of α.

For a curve in general, the change is one of the following moves, up to the choice of overcrossings or undercrossings. (Fig. 44)



M.1 M.2 M.3

Fig. 44

These moves are the operations $\Omega.1$, $\Omega.2$, $\Omega.3$ and their inverse operations $\Omega'.i$ ($i = 1, 2, 3$) in [18]. We denote $\Omega.i$ and $\Omega'.i$ by M.i ($i = 1, 2, 3$). Any projection of a knot can be obtained from a given projection of the knot by performing a sequence of such operations on the original projection [18].

The moves M.2 and M.3 do not change the kinking number. A loop as in M.1 with kinking number +1 cannot occur for any $\alpha > 0$, which follows from Stokes' theorem as in §4. From Stokes' theorem also follows that a loop with kinking number -1 cannot appear or disappear when α increases, except in the move M.2, where the kinking number remains the same anyhow. (Fig. 45)

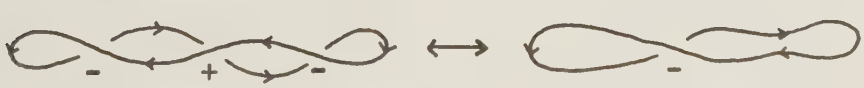


Fig. 45

So $k(\alpha)$ is constant for $0 < \alpha \leq 90^\circ$. It follows now from (1) that $k(\alpha) = k(0) - b$, $0 < \alpha \leq 90^\circ$. Q.E.D.

EXAMPLE In §5 we embedded knot 5_2 as an integral curve with kinking number +1. As an illustration we shall now embed this knot in the positive twist projection.

Consider the projection of 5_2 in figure 46. It is the original projection in §5, fig. 32, rotated 90° counterclockwise. We have also given it spikes.

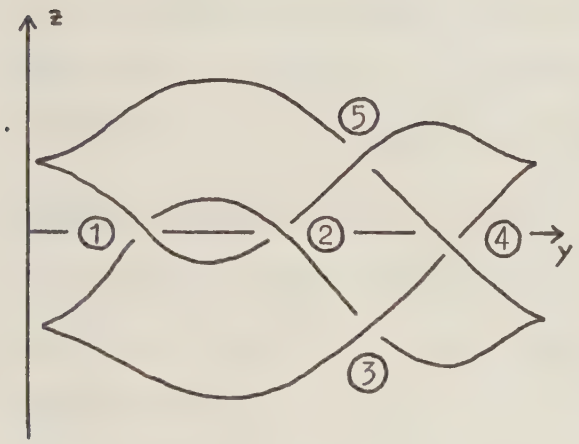


Fig. 46

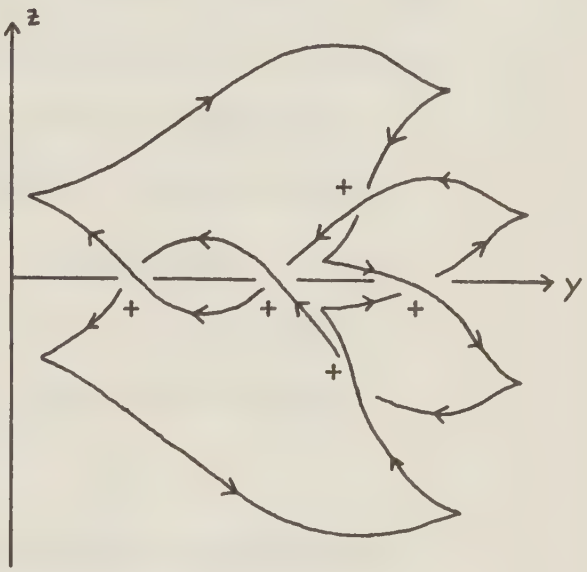


Fig. 47

The crossings ① , ② and ④ have positive twists, but ③ and ⑤ do not, so we have to modify these two crossings. This has been done in figure 47. We then get two more spikes pointing to the right (and also two more spikes pointing to the left). We now have an integral curve in the positive twist projection. Its kinking number in the kinking number projection is 5 - (number of spikes pointing to the right) = 5 - 4 = +1.

§7. Knotting criteria

We found in §3, that if the standard and non-standard contact structures were equivalent, then we should be able to construct an integral curve which bounds an embedded disc, and which has kinking number 0 in the standard structure. By adding an extra loop to the integral curve in figure 34 in §5, we would of course get an integral curve with kinking number 0. However, this curve is a knot, so it cannot bound an embedded disc. In Chapter 3 we shall prove the following knotting criterion.

THEOREM (Integral curve version) A closed integral curve of the standard contact structure $xdy + dz = 0$, with kinking number ≥ 0, is a knot, that is such a curve cannot bound an embedded disc.

From this theorem follows that the standard and non-standard contact structures are non-equivalent.

REMARK The knotting criterion is valid in any standard contact structure, if we use a suitable definition of kinking number. We shall introduce such a description in Chapter 3, §5. The kinking number of a closed curve γ in space is the number of times an orientable surface with boundary γ twists around γ.

We can also formulate the theorem above without any reference to contact geometry at all. This follows from what we found in the preceding section on the positive twist projection. We shall now explain how it works.

Fix a direction in space, for example the y-axis in a coordinate system. Consider a projection of a closed curve, all of whose crossings have positive twists with respect to the y-axis. Let h be a function which gives the y-coordinate of the embedded circle. The function h is called the height function. The curve segments around the relative maxima (or relative minima) of h are called bridges [20]. (Fig. 48)

Fig. 48

Let b be the number of bridges. Let $\mathscr{k}$ be the kinking number of the curve in the given projection. We then have the following version of the theorem above.

THEOREM (Positive twist version) A closed curve with only positive twists and $\mathscr{k} \geq b$ is knotted.

That the two versions are equivalent follows from §6. We just round off the spikes in the positive twist projection, and note that each spike pointing to the right then contributes a bridge. We found in §6 that $\mathscr{k} - b = k$, where k is the kinking number in the kinking number projection. It follows that $\mathscr{k} \geq b$ if and only if $k \geq 0$.

So we consider an integral curve in the standard structure $xdy + dz = 0$, with kinking number ≥ 0, and we want to prove that the curve is a knot. One difficulty in proving such a theorem is that the projection in the xy-plane is arbitrary. The only knotting criterion of this kind known so far is the Alternating Knot Theorem [3], [14], [15].

It states that a circle which admits an alternating projection is knotted, except in certain trivial cases. A projection is alternating when the curve successively passes over and under itself. Figure 49 illustrates an alternating knot (10_{84} in [19, p. 407]). A trivial unknotted alternating projection is given in figure 50.

Fig. 49

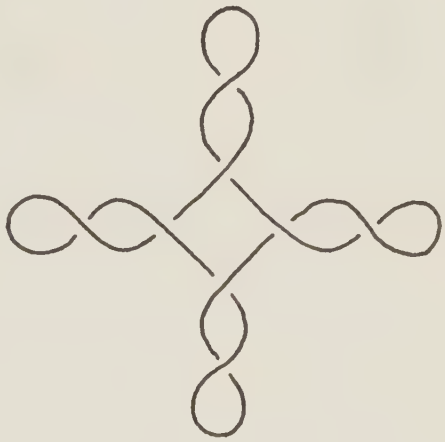


Fig. 50

What is proved is that the Alexander polynomial of an alternating circle is non-trivial, and therefore the curve must be knotted.

The Alexander polynomial can be computed for any knot. However, there are infinitely many knots for which the polynomial is trivial [6], [19, pp. 166-167, 173], [21]. In these cases the polynomial cannot be used to distinguish the knot from the unknotted circle, which also has trivial polynomial.

In §8 we shall describe a knotted integral curve with trivial Alexander polynomial and kinking number +1. This example shows that in order to prove our theorem, we have to find a method which does not depend on the Alexander polynomial. We shall develop such a method in Chapter 3.

§8. An important example

We shall give an example of an integral curve with kinking number +1 and trivial Alexander polynomial. We shall first discuss the curve in the kinking number projection. The doubled knot in figure 51 has trivial polynomial [19, p. 167]. We move two half twists to the bottom of the knot. (Fig. 52) We do this in order to get a projection which can be modified to an integral curve with kinking number +1. The original projection gives an integral curve with kinking number ≤ 0.

Fig. 51

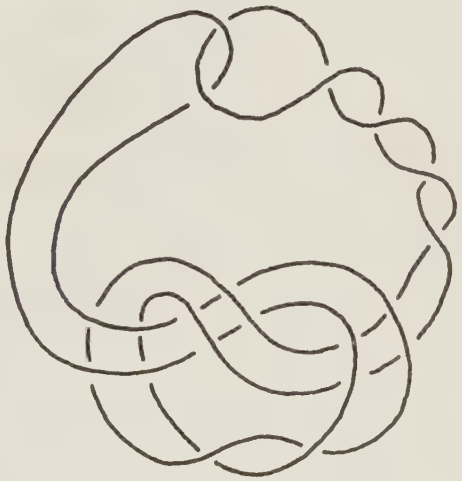


Fig. 52

We shall now modify the projection in figure 52 to an integral curve. By using the colour method described in §5, we find that we need at least seven correction loops, so the kinking number is $\leq +1$. (Fig. 53)

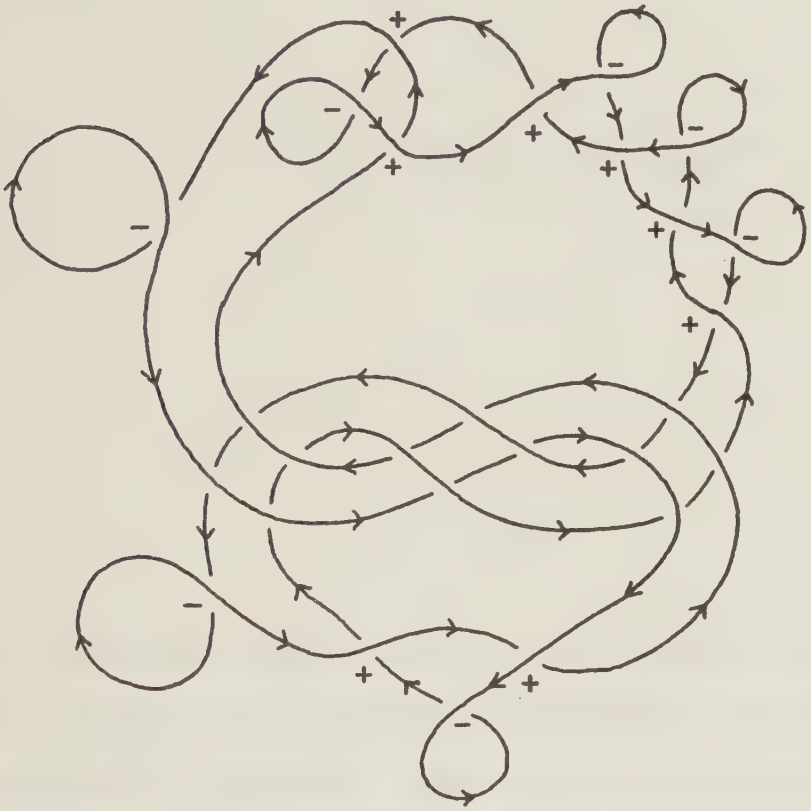


Fig. 53

We shall show that the loops indicated in figure 53 are sufficient. As the original knot has twenty crossings, we choose not to solve the system of inequalities described in §5, but instead we construct the knot in the positive twist projection.

The center knot is a trefoil knot, so we first construct this knot in the positive twist projection. (Fig. 54)

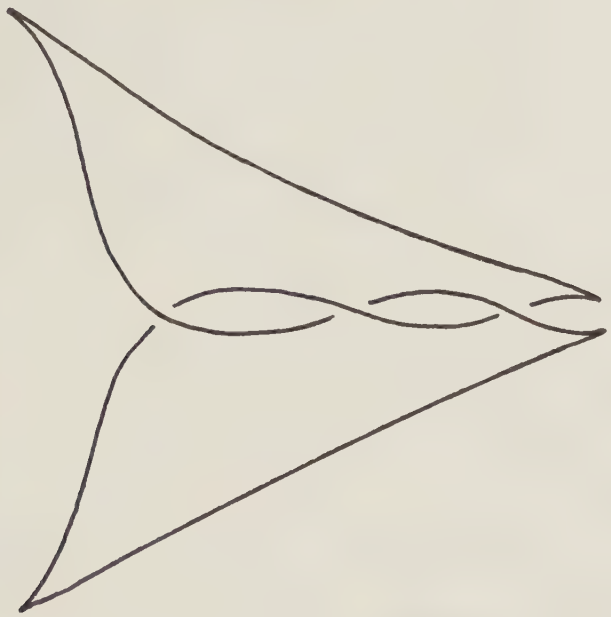


Fig. 54

Then we arrange the doubled knot around the trefoil, and we obtain a positive twist projection of the doubled knot with kinking number +8 and bridge number 7, as shown in figure 55 on the next page. The kinking number in the kinking number projection is therefore 8 - 7 = +1. Remember that each spike pointing to the right contributes -1 to the kinking number in the kinking number projection.

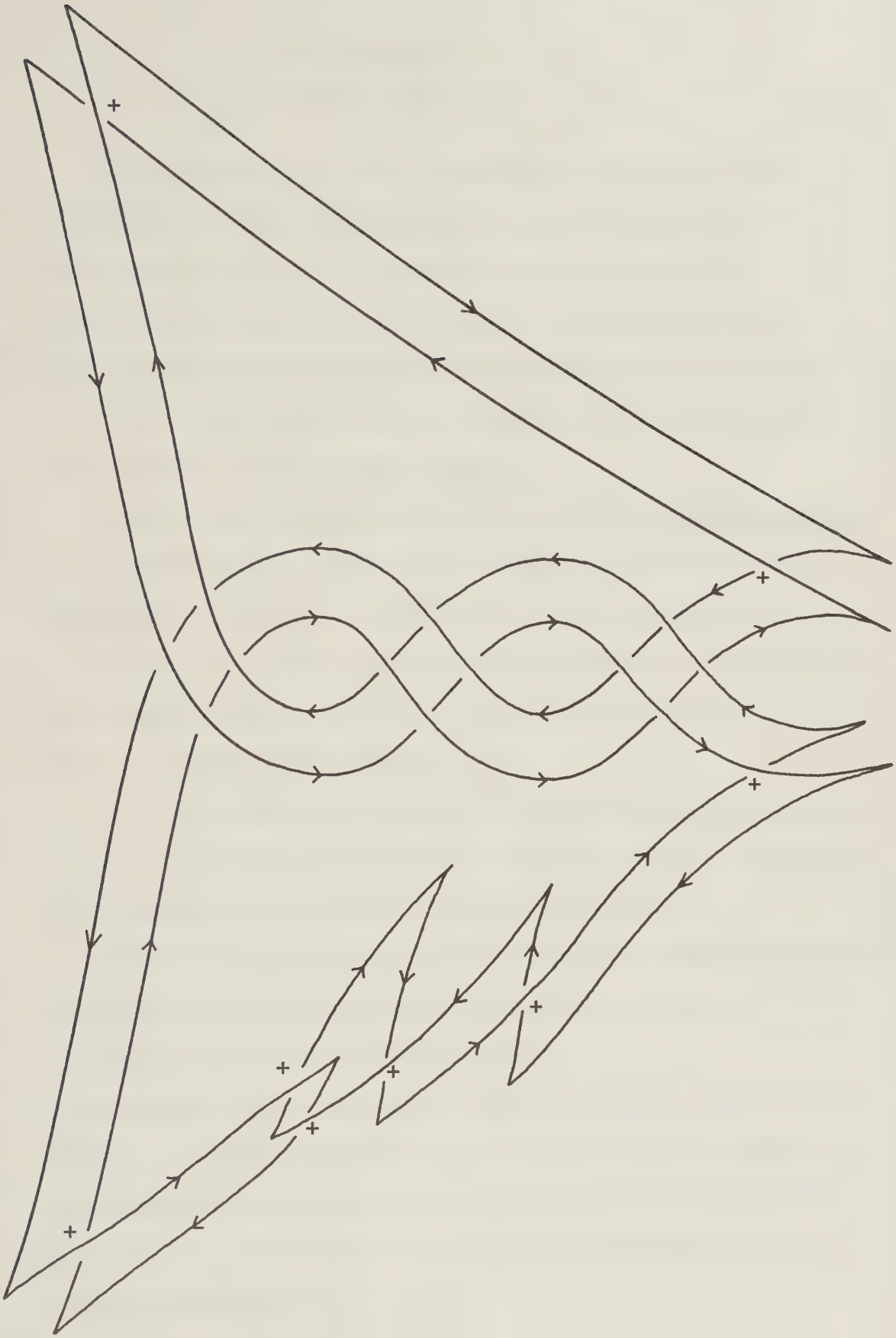


Fig. 55

CHAPTER 3

GEOMETRY OF THE DISC

In this chapter we prove the knotting criterion we stated and discussed in the Introduction, §6, and in Chapter 2, §7. We do this by studying the geometry of the disc. From this criterion, and the discussion in Chapter 2, follows that the standard and non-standard contact structures are non-equivalent.

In §1 we describe a convenient presentation of an embedded disc in space, the bridge presentation.

In §2 we introduce and discuss the so called movie presentation of an embedded disc, in which the embedding is described by the intersection patterns in a sequence of planes intersecting the disc.

In §3 we study the different types of saddles a disc can have, and we describe them in the movie presentation. The basic type of saddle is the arc-to-arc saddle.

In §4 we introduce a fundamental concept, the parameter disc, in which we determine the number of arc-to-arc saddles. If the disc has b bridges, we find it has b-1 arc-to-arc saddles.

In §5 we give a geometric description of the kinking number as the number of times the disc twists around its boundary.

In §6 we prove our knotting criterion.

With minor modifications our discussion above is valid also for embedded, orientable surfaces of higher genus with one boundary component, which is an unknotted or knotted circle. In an Appendix we discuss this, and prove the Genus Estimation Theorem we stated in the Introduction, §6.

§1. Bridge presentation of a disc

Consider an embedded disc D in space with boundary γ. Introduce a coordinate system in space, and let h be a function which gives the y-coordinate of D. We call h the height function of the disc.

Assume that all relative maxima of the height of the boundary γ have the same y-coordinate, y_1, and that all relative minima have the same y-coordinate, y_0, $y_0 < y_1$. This can always be achieved by stretching the disc. Note that the height of γ has the same number of maxima and minima.

The arc segments around the maxima (or the minima) are called bridges [20]. (Fig. 56)

Fig. 56

We say that the disc has bridge number b if the height of the boundary γ has b maxima. The maximum and minimum points of the height of γ will be called bridge points.

We have now obtained what we call a bridge presentation of the disc. If the disc has bridge number b, then the boundary of the disc consists of 2b bridge points and of 2b arcs connecting them.

§2. Movie presentation of a disc

Assume a disc has a bridge presentation with bridge number b, and height function in the direction of the y-axis. We assume that the boundary γ of the disc is oriented, and that the height of γ has its minima at y_0 and its maxima at y_1.

Now consider a continuous sequence of planes orthogonal to the y-axis, and all intersecting the disc. We call such a sequence a movie. Consider a plane in the movie. The 2b arcs of the boundary of the disc penetrate the plane in 2b points. We assign a + to a point of penetration in the plane, if an insect, which follows the orientation of the arc, moves in the positive y-direction. We assign a - otherwise. We get b points with a +, and b points with a -. (Fig. 57)

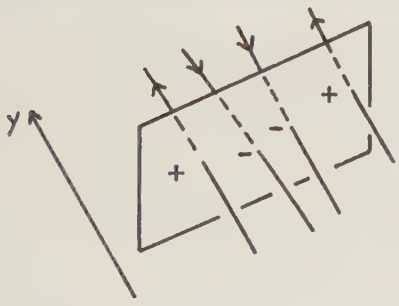


Fig. 57

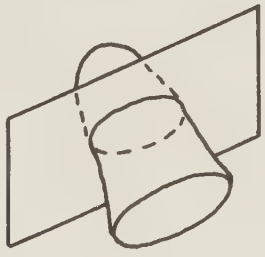


Fig. 58

The intersection of the disc with a plane in the movie consists of b arcs and a finite number of simple closed curves. These arcs and curves do not intersect each other, except in a finite number of planes. We shall discuss these cases in the next section.

REMARK A simple closed curve in a movie plane is the intersection between the plane and a bag of the disc. (Fig. 58)

An intersection arc in a movie plane connects a point of penetration labelled - with a point labelled +. We give the arc an orientation from - to +. (Fig. 59)

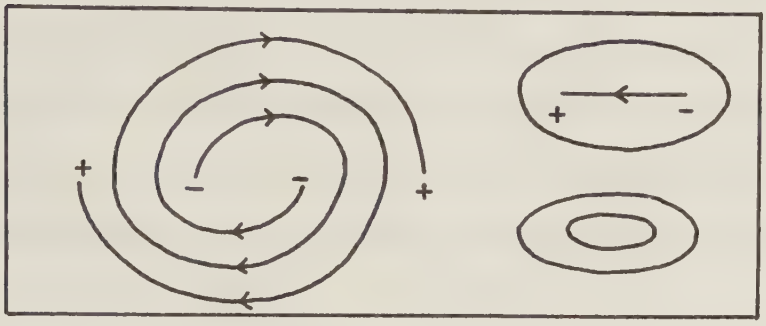


Fig. 59

At $y_0 + \varepsilon$ and $y_1 - \varepsilon$, where $\varepsilon > 0$ is small, the intersection of the disc with a movie plane consists of short straight arcs and simple closed curves in the plane. (Fig. 60)

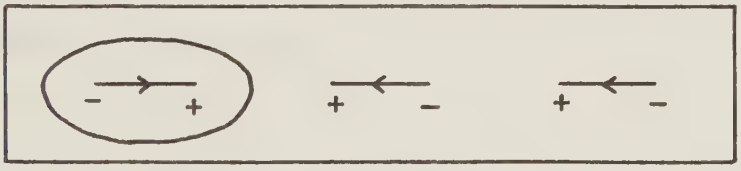


Fig. 60

The intersection patterns in the movie planes describe the embedding of the disc. We call such a description of the disc a movie presentation.

§3. Saddles in a movie presentation of a disc

Consider a movie presentation of a disc. The pattern of intersection arcs and simple closed curves in the planes changes essentially only in a finite number of planes in the movie. By an essential change we mean a change of pattern, which cannot be obtained by just moving the intersection curves around in the 2b-punctured planes. At an essential change, arcs and closed curves come together and intersect each other, forming a so called saddle.

There are three types of saddles, which we shall now describe.

Arc-to-arc saddles An arc-to-arc saddle is a change of pattern of two arcs in the movie planes. At an arc-to-arc saddle, two arcs come together and then come apart again, forming a different pattern. The orientations of the arcs should be compatible during the process. We illustrate such a saddle in figure 61. Note that the procedure is reversible.

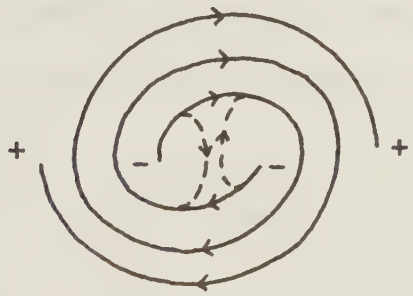


Fig. 61

Circle-to-arc saddles At a circle-to-arc saddle a simple closed curve and an arc come together, forming an arc, or an arc splits into a simple closed curve and an arc. (Fig. 62)

Fig. 62

Circle-to-circle saddles At a circle-to-circle saddle two simple closed curves come together, forming one simple closed curve, or one simple closed curve splits into two simple closed curves. (Fig. 63)

Fig. 63

REMARK A circle-to-circle saddle in the movie planes is the intersection between the planes and a pair of pants of the disc. (Fig. 64)

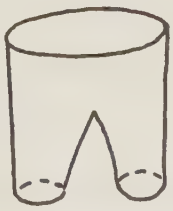


Fig. 64

§4. The parameter disc

Consider a bridge presentation of a disc with bridge number b. We assume the boundary is oriented. The planes in a movie intersect the disc along curves. These curves fill up the disc, and is called a foliation with singularities of the disc. We can describe this foliation in a special disc, which we call the parameter disc.

We describe the situation in an example. We have a bridge presentation in fig. 65, and the corresponding parameter disc in fig. 66.

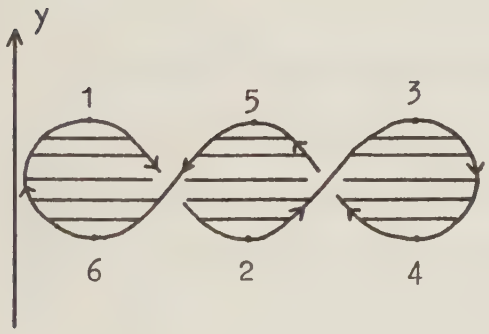


Fig. 65

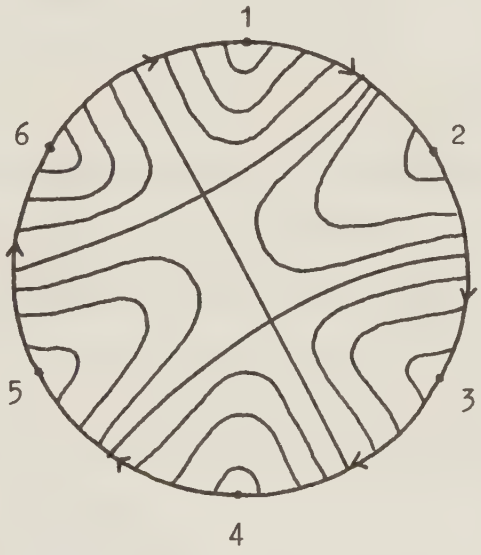
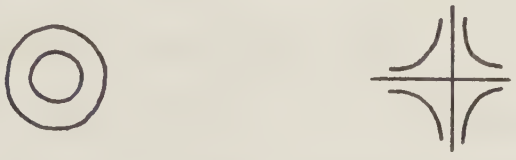


Fig. 66

When we follow the orientation around the boundary of the embedded disc, we could start at a bridge point of maximum height, say at the point marked 1 in fig. 65. We then come to the minimum point 2, and so on. We mark these points on the boundary of the parameter disc, and give its boundary the corresponding orientation. The points 1, 2,..., will be called bridge points also on the parameter disc. The foliation curves are transversal to the boundary of the parameter disc, except at the 2b bridge points, where we have a singularity.

We always assume that the embedded disc is in Morse regular position with respect to the movie, i.e. the height function has isolated, non-degenerate critical points (see for example [13]).

The singularities of the foliation are of two types, centers and saddles. (Fig. 67)



Center Saddle

Fig. 67

The saddles are of three types as we described in the preceding section. We have illustrated a foliation of the parameter disc in fig. 68. The arc-to-arc saddles are marked A-A, the circle-to-arc saddles C-A, and the circle-to-circle saddle is marked C-C.

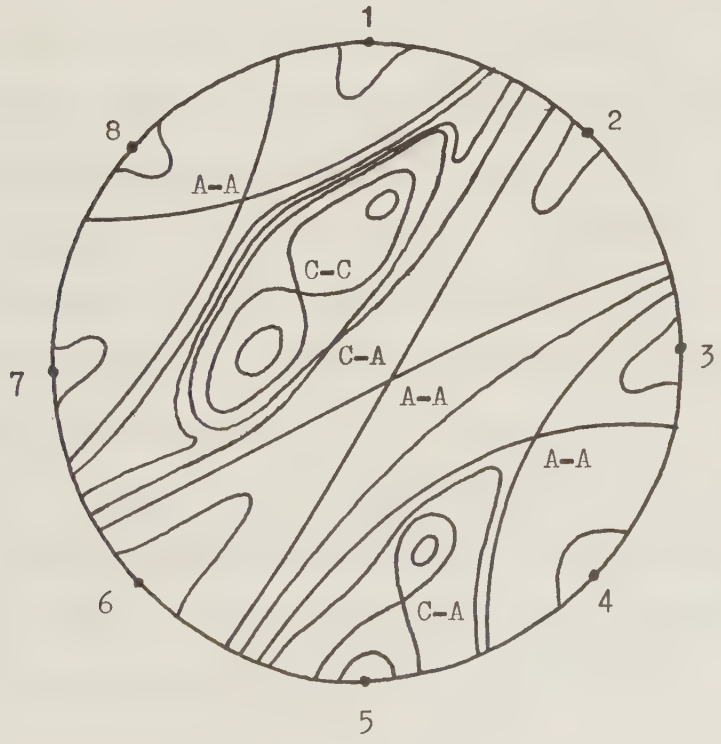


Fig. 68

The example in fig. 68 has eight bridge points, i.e. b = 4. We have three arc-to-arc saddles. In the general case we have the following result.

THEOREM A foliation of the parameter disc with 2b bridge points has precisely b-1 arc-to-arc saddles.

Proof: Assume the parameter disc is the northern hemisphere of a sphere. We extend the foliation of the parameter disc to the southern hemisphere, so we get a foliation of the sphere, such that the patterns are identical on the northern and southern hemispheres.

Let C be the number of centers in a foliation, and S the number of saddles. On the sphere we have the formula by Poincaré [16],

$$S = C - 2.$$

We always have a center at each of the 2b bridge points. We now assume that we have no more centers in the foliation of the sphere. The number of saddles on the sphere is then 2b-2, i.e. we have b-1 saddles on the parameter disc. These saddles are arc-to-arc saddles.

We now assume that the foliation of the parameter disc also has circle-to-arc saddles and circle-to-circle saddles. These singularities consist of concentric circles. We can step-by-step shrink each such family to a point, as described for example in [7, p. 20]. What remains is a parameter disc with 2b bridge points, and whose only singularities besides the bridge points are the arc-to-arc saddles of the original foliation. It follows that there are b-1 arc-to-arc saddles also in the general case. Q.E.D.

§5. A geometric description of the kinking number

In Chapter 2, §3, we introduced an arithmetic definition of the kinking number of a regular projection of a closed curve. In this section we give a geometric description of the kinking number of an unknotted curve, as the number of times an embedded disc twists around its boundary. In a final remark we show how to describe the kinking number of any closed curve γ, as the number of times an embedded, orientable surface with boundary γ twists around γ.

a) Consider a bridge presentation of a disc with bridge number b, with height function along the y-axis in a coordinate system, and whose boundary γ has kinking number $\varkappa$ in a regular projection from $+\infty$ along the x-axis. The curve γ can be any unknotted closed curve, it is not supposed to be an integral curve. We assume the height of γ has its maxima at y_1 and its minima at y_0. (Fig. 69)

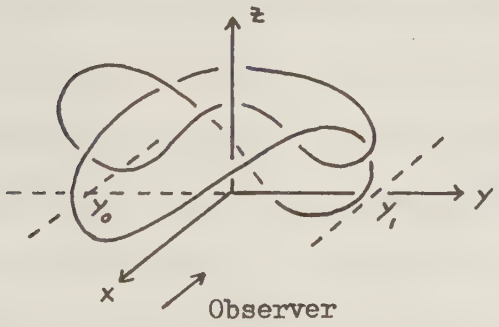


Fig. 69

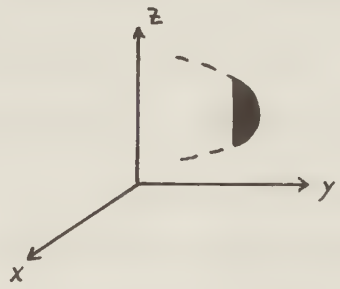


Fig. 70

In the intervals $[y_0, y_0 + \varepsilon]$ and $[y_1 - \varepsilon, y_1]$, $\varepsilon > 0$ small, we assume the disc is embedded parallel to the yz-plane in the way we have illustrated in figure 70. Any embedding of a disc can be modified to fulfil this condition.

Consider along γ a narrow band N which is part of the disc. The band contains at each point of γ a vector t transversal to γ. At each point of γ, except at the 2b bridge points, we can assume that t is in the plane ortogonal to the y-axis. The vector t is parallel to the z-axis near the bridge points, especially at $y_0 + \varepsilon$ and $y_1 - \varepsilon$, and two vectors near the same bridge point have opposite directions. (Fig. 71)

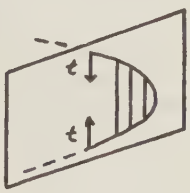


Fig. 71

We shall now define the twist τ of the disc around γ. We assume γ is oriented. The twist will be independent of the orientation of γ, but it is convenient to introduce an orientation in order to describe the twist. Consider a point S on γ with coordinate $y_0 + \varepsilon$, and assume the orientation of γ at S is in the positive y-direction. S is initial point of the associated vector t we defined above. We now move the plane through S, which is orthogonal to the y-axis, in the positive y-direction. The vector t will then rotate around γ in the plane, and we consider the rotation of t from a point behind the plane. (Fig. 72)

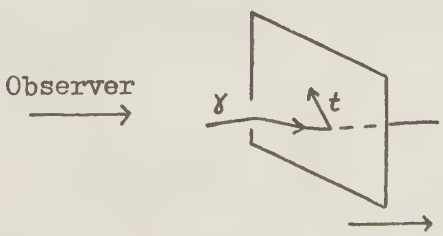


Fig. 72

We let counterclockwise rotation be positive, and we let $R_1(t)$ be the net rotation between $y_0 + \varepsilon$ and $y_1 - \varepsilon$. When we have reached $y_1 - \varepsilon$, we pass the bridge point and come back to $y_1 - \varepsilon$ at a point T on γ. We then move the plane in the negative y-direction, and we consider the rotation of the vector t from a point behind the plane, as described in figure 73.

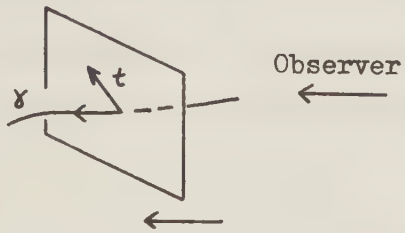


Fig. 73

We let counterclockwise rotation be positive as before, and we let $R_2(t)$ be the net rotation between $y_1 - \varepsilon$ and $y_0 + \varepsilon$. We do this procedure 2b times, once for each arc γ_i in the bridge presentation, and we let $R_i(t)$ be the corresponding net rotation of t for γ_i. If the minima (and the maxima) of the height of γ have different y-coordinates, we measure $R_i(t)$ for each γ_i between $y^i_{min} + \varepsilon$ and $y^i_{max} - \varepsilon$ as before. We now give the following definition.

DEFINITION $\sum R_i(t)$ is the twist τ of an embedded disc around its boundary. τ depends on the projection, and the disc should be orthogonal to the direction of observation in a neighbourhood of the bridge points. Any embedding of a disc can be modified to fulfil this condition.

The twist τ of a disc is related to the kinking number $\mathcal{K}$ of its boundary. We have the following important result.

PROPOSITION Let τ be the twist of a disc around its boundary γ, and $\mathcal{H}$ the kinking number of γ, where $\mathcal{H}$ and τ are defined with respect to the same projection. We then have

$$\mathcal{H} = \tau.$$

Proof: The band N we used in the description of τ is part of the disc, so the boundaries γ and γ' of N are unlinked. This is only possible if their linking number is 0. However, the linking number of γ and γ' is $\mathcal{H} - \tau$ with the sign conventions we have adopted for $\mathcal{H}$ and τ. (Fig. 74)

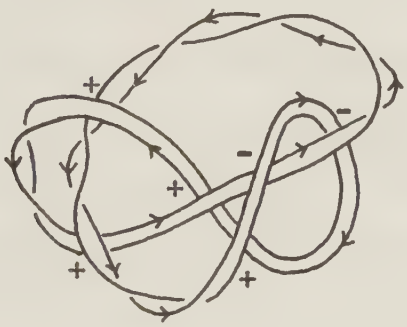


Fig. 74

So it follows that $\mathcal{H} - \tau = 0$, i.e. $\mathcal{H} = \tau$. Q.E.D.

b) We shall now describe a somewhat different method to find the twist τ. Consider the same bridge presentation of a disc as in a). It is possible to find the twist τ by using only one movie sequence from $y_0 + \varepsilon$ to $y_1 - \varepsilon$. Consider a plane orthogonal to the y-axis with y-coordinate between $y_0 + \varepsilon$ and $y_1 - \varepsilon$. The plane intersects γ in 2b points P_i, where the 2b arcs of γ in the bridge presentation intersect the plane. Each point P_i is initial point of the vector t_i described in a).

We assume we have assigned a + or a - to each point P_i as we described in §2, according as an insect, which follows the orientation of an arc, moves in the positive or negative y-direction, respectively. When we move the plane from $y_0 + \varepsilon$ to $y_1 - \varepsilon$, each vector t_i rotates around the point P_i in the plane. We consider the rotation from a point behind the plane in the same way as before. We let counterclockwise rotation be positive if P_i is assigned a + and negative if P_i is assigned a -. We let the net rotation of t_i around P_i be $R_{P_i}(t_i)$. It follows that $\sum R_{P_i}(t_i)$ is the twist τ of the disc around γ as in the definition above. When the minima (and the maxima) of the height of γ have different y-coordinates, we can find τ in the following way. Intersection points then appear and disappear when we move the plane along the y-axis. We measure the net rotation of t_i between $y^i_{min} + \varepsilon$ and $y^i_{max} - \varepsilon$ as before, and the twist τ is the sum of these contributions.

c) We shall now define the twist of the disc associated to an intersection arc between the disc and a plane orthogonal to the y-axis. Consider an oriented intersection arc α between the disc and the plane at y orthogonal to the y-axis. We choose the orientation of α as in §2, i.e. from the point P assigned a - to the point Q assigned a +. Let $r(\alpha)$ be the number of times a tangent to α rotates, when the tangent is moved from P to Q. We consider the rotation from a point with coordinate y' where $y' < y$, and we let counterclockwise rotation be positive.

DEFINITION The number $r(\alpha)$ is the twist of the disc associated to an intersection arc α between the disc and a plane orthogonal to the y-axis.

The twist $r(\alpha)$ associated to an arc α at y is the sum of the net rotations of the vectors t around P and Q between $y_0 + \varepsilon$ and y, provided α is not part of a saddle somewhere between $y_0 + \varepsilon$ and y. (Fig. 75)

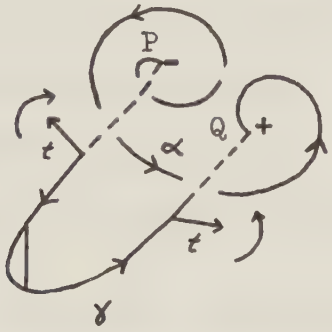


Fig. 75

REMARK We assumed above that the band N is part of a disc. However, the boundary curves of N are unlinked also when N is part of any orientable, embedded surface with boundary γ. So the surface twists in the same way also in that case, and the twist τ equals the kinking number $\mathcal{K}$ of the boundary γ as in the Proposition above. In the Appendix on p. 81 we use this observation to prove a Genus Estimation Theorem for orientable, embedded surfaces with boundary.

§6. Proof of the knotting criterion

We shall prove the following knotting criterion, which we stated and discussed in the Introduction, §6, and in Chapter 2, §7.

THEOREM (Positive twist version) Consider a closed curve embedded in space with kinking number $\mathcal{K}$ and with b bridges. Assume all crossings have positive twists. If $\mathcal{K} \geq b$, then the curve is knotted, i.e. it cannot bound an embedded disc.

This theorem is equivalent to the Integral Curve Version of the knotting criterion in Chapter 2, §7. From the Integral Curve Version follows that the standard and non-standard contact structures are non-equivalent.

Proof of the theorem: We consider a bridge presentation of a disc with b bridges, with height function along the y-axis in a coordinate system, and whose boundary γ has kinking number $\mathcal{K}$ in a regular projection from $+\infty$ along the x-axis. We assume the curve γ has crossings with positive twists with respect to the y-direction. We shall show that $\mathcal{K} < b$, i.e. if $\mathcal{K} \geq b$, then γ cannot bound an embedded disc.

We can assume that the height of γ has its maxima at y_1 and its minima at y_0, and that the disc is embedded parallel to the yz-plane in a neighbourhood of the bridge points (Compare figure 69 and figure 70 in §5). We shall later in the proof make some further assumptions concerning the embedding of the disc.

The 1-bridge case For a 1-bridge presentation we have $\mathcal{H} < 1$ when the crossings have positive twists. (Fig. 76)

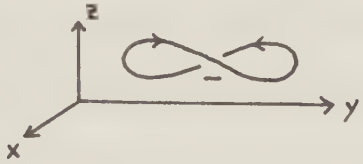


Fig. 76

The b-bridge case We found in §4 that we have precisely b-1 arc-to-arc saddles, and we have a finite number of circle-to-arc and circle-to-circle saddles. By slightly deforming the disc we can assume that all saddles have different height. We cut the disc along the b-1 arc-to-arc saddles. The disc is then decomposed into strips, i.e. 1-bridge discs. The strips are parallel to the y-axis, and their contribution s to the kinking number of γ satisfies $s \leq 0$. This follows from the fact that the two boundary components of the strip, which are parts of γ, have opposite orientations, as the disc is an orientable surface, and that all crossings of γ have positive twists. (Fig. 77)

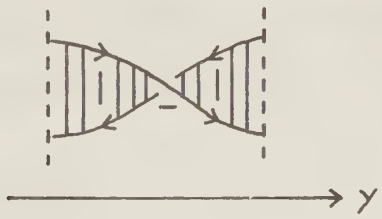


Fig. 77

We let τ be the twist of the disc around γ, as we defined in §5a. From the Proposition in §5a we have $\mathcal{H} = \tau$. We must show that an arc-to-arc saddle contributes at most one unit to the twist τ. As we have b-1 arc-to-arc saddles, and the contribution s from the strips satisfies $s \leq 0$, it would follow that $\mathcal{H} \leq s + (b-1) \leq b-1$.

In the movie presentation an arc-to-arc saddle is a change of pattern in the movie planes, which in our case are orthogonal to the y-axis. Let α_1 and α_2 be the two arcs slightly below the saddle, β_1 and β_2 the two arcs slightly above the saddle. The arcs are oriented as described in §2. The arcs α_1 and α_2 come together at a point S, and then come apart again, forming two new arcs, β_1 and β_2. We assume that the arcs are parallel to the xy-plane near the endpoints. This can be achieved by a slight deformation of the disc. (Fig. 78)

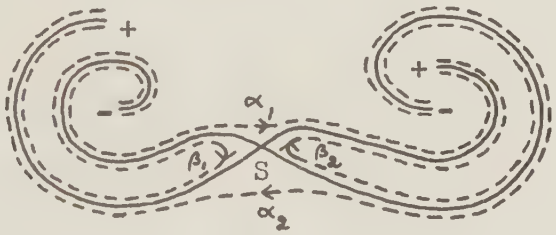


Fig. 78

In a small neighbourhood U of S we can describe the saddle in the following way. Let σ be a straight arc through S, which connects α_1 and α_2. The arcs α_1 and α_2 are connected by two oriented arcs $\sigma_1 = A_1B_1$ and $\sigma_2 = B_2A_2$ close to σ, the orientations of σ_1 and σ_2 being compatible with the orientations of α_1 and α_2. (Fig. 79)

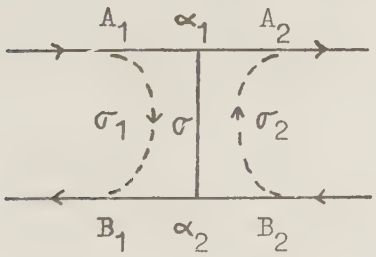


Fig. 79

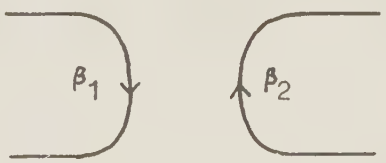


Fig. 80

As α_1 and α_2 are close to β_1 and β_2 outside U, we can consider σ_1 and σ_2 as part of β_1 and β_2, resp. (Fig. 80) By a small deformation of the disc we can assume that α_1, α_2, β_1 and β_2 are parallel to the xy-plane near the points A_1, A_2, B_1 and B_2.

We shall now determine how much an arc-to-arc saddle contributes to the twist τ of the disc around γ. In §5c we defined the twist $r(\alpha)$ of the disc associated to an arc α. Let $r(\alpha_i)$ be the twist of the disc associated to the arc α_i, and $r(\beta_i)$ the twist of the disc associated to the arc β_i $(i = 1, 2)$. Let $r_\alpha = r(\alpha_1) + r(\alpha_2)$ and $r_\beta = r(\beta_1) + r(\beta_2)$. The contribution of the saddle to the twist of the disc around γ is denoted by τ_{saddle}, and

$$\tau_{saddle} = r_\alpha - r_\beta. \tag{1}$$

Let $r(\sigma_i)$ $(i = 1, 2)$ be the number of times a tangent to σ_i rotates, when the tangent is moved along σ_i in the direction of the orientation of σ_i. We choose the same sign convention for $r(\sigma_i)$ as for $r(\alpha_i)$ and $r(\beta_i)$, $i = 1, 2$. The arcs α_1 and α_2 are close to β_1 and β_2 outside the neighbourhood U of S. The contributions to r_α and r_β are therefore the same except along σ_1 and σ_2, and we get the formula

$$\tau_{saddle} = -(r(\sigma_1) + r(\sigma_2)). \tag{2}$$

We have $r(\sigma_1) + r(\sigma_2) = \pm 1$, where the sign depends on the orientation of the saddle. So $\tau_{saddle} = \pm 1$, and it follows that the contribution to the kinking number from the b-1 arc-to-arc saddles is at most b-1. Therefore the kinking number of the disc satisfies $\mathcal{K} \leq b-1$. Q.E.D.

APPENDIX

A GENUS ESTIMATION THEOREM

In the Introduction, §6, we stated the following result, which we found during our study of knots in the standard contact structure.

GENUS ESTIMATION THEOREM Let γ be a closed curve embedded in space with kinking number $\varkappa$ and with b bridges. Assume all crossings have positive twists. Let g be the genus of an orientable, embedded surface with boundary γ. Then

$$2g \geq \varkappa - b + 1.$$

REMARK If γ is considered as an integral curve with kinking number $k = \varkappa - b$ as in Chapter 2, we get the inequality

$$k \leq -1 + 2g.$$

For a disc ($g = 0$) we get $k \leq -1$.

We now observe that with minor modifications, which we shall describe below, our discussion in Chapter 3 is valid not only for the disc, but for any orientable surface with boundary.

We assume an orientable, embedded surface of genus g has one boundary component γ, which is an unknotted or knotted circle. We can as before describe the embedded surface in a bridge presentation and in a movie presentation. The saddle types are the same as for the disc, i.e. we have arc-to-arc saddles, circle-to-arc saddles and circle-to-circle saddles.

As we remarked in Chapter 3, §3, for discs a circle in a movie plane is the intersection between the plane and a bag of the disc. However, surfaces of genus $g > 0$ do not only have bags, but can also have <u>tubes</u> as illustrated in figure 81.

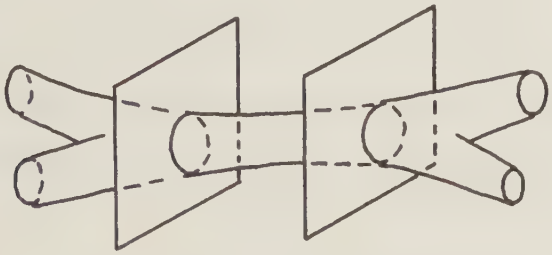


Fig. 81

So for a surface of genus $g > 0$, a circle in a movie plane can be the intersection between a bag and the plane, or be the intersection between a tube and the plane.

The parameter disc is replaced by the <u>parameter surface</u> with boundary γ. We can describe the parameter surface as in figure 82, which illustrates the case $g = 2$.

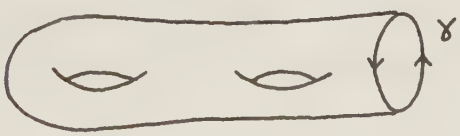


Fig. 82

The embedded surface and the parameter surface are foliated by the movie planes as before. As for the disc we shall assume that the surface is in Morse regular position with respect to the movie.

We have the following result for the number of arc-to-arc saddles.

THEOREM Let γ be a closed curve embedded in space with b bridges. Let S_A be the number of arc-to-arc saddles of an orientable surface of genus g, embedded in space with boundary γ. We then have

$$b - 1 \leq S_A \leq b - 1 + 2g.$$

Proof: Glue another surface of genus g onto the parameter surface along the boundary γ, so we obtain a closed surface of genus G = 2g. (Fig. 83)

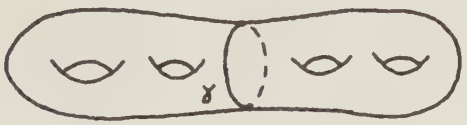


Fig. 83

Extend the foliation of the parameter surface to the whole surface, so that the patterns are identical on the two parts. Each bridge point then contributes a center. We assume there are no other centers. Let C be the number of centers, and S the number of saddles in a foliation of a closed, orientable surface of genus G. We then have the following formula by Poincare' [17],

$$S = C - 2 + 2G.$$

As C = 2b and G = 2g in our case, we get S = 2b - 2 + 4g on the closed surface, i.e. we have b - 1 + 2g saddles on the parameter surface. These saddles are arc-to-arc saddles.

We now assume that we also have circle-to-arc saddles and circle-to-circle saddles on the parameter surface. We remove these saddles as for the disc. However, the new surface can be of lower genus than the original surface, as we may have removed handles in the process. We have illustrated such a case in figure 84, where we have a 1-bridge presentation of a circle γ, which is the boundary of a surface of genus two. There are circle-to-arc and circle-to-circle saddles in this case, but no arc-to-arc saddles.

Fig. 84

Let g_1 be the genus of the new surface. This surface has the same number of arc-to-arc saddles as the original surface, and no other singularities except the 2b bridge points. From the first half of the proof follows that the number of arc-to-arc saddles satifies

$$S_A = b - 1 + 2g_1.$$

As $0 \leq g_1 \leq g$, we have the inequalities

$$b - 1 \leq S_A \leq b - 1 + 2g.$$

Q.E.D.

REMARK Figure 84 on the preceding page illustrates a case where the number of arc-to-arc saddles is b-1. In figure 85 we have illustrated a case where the number of arc-to-arc saddles is $b - 1 + 2g$. The surface has genus $g = 2$, bridge number $b = 2$, and has $2 - 1 + 4 = 5$ arc-to-arc saddles.

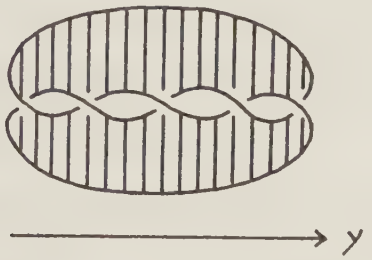


Fig. 85

Finally we consider the kinking number. As we remarked in Chapter 3, §5, the kinking number of a closed curve γ is the number of times an orientable, embedded surface with boundary γ twists around γ, where we have modified the embedding as for the disc.

Proof of the Genus Estimation Theorem: The proof now follows in the same way as for the disc. We cut the surface along the arc-to-arc saddles. The 1-bridge pieces contribute ≤ 0 to the kinking number, and each arc-to-arc saddle +1 or -1. As we have at most $b - 1 + 2g$ arc-to-arc saddles it follows that $\kappa \leq b - 1 + 2g$. This inequality can also be written as $2g \geq \kappa - b + 1$. Q.E.D.

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